STUDIES IN COSMOLOGICAL DYNAMICS

by

WALDIR LEITE ROQUE

Being a dissertation submitted to the Faculty of Science of the University of Cape Town in the fulfilment of the requirements for the degree of Doctor in Philosophy in Applied Mathematics.

University of Cape Town

1985

DEDICATION

'A MINHA ESPOSA, ANA

TO MY WIFE, ANA

E AOS MEUS FILHOS, BRENO E BRUNO

AND TO MY SONS, BRENO AND BRUNO'

'TO ALL THOSE PEOPLE WHO HAVE BEEN OPPRESSED

BY THE APARTHEID SYSTEM'

ACKNOWLEDGEMENTS

I wish to express sincere gratitude to my supervisor, Professor George Ellis, both for suggesting the topics addressed in this thesis and for his constant encouragement. I feel honoured to have worked with him.

Thanks to Professor David Matravers for many general discussions on cosmology, and for the generous hospitality I have received from him personally and the members of his staff in the Applied Mathematics Department at UCT. Ian MacArthur and his wife were especially kind.

The members of the Cosmology Group deserve special mention. In particular,

- Professor W. B. Bonnor for fruitful discussions on inhomogeneous cosmologies.
- Mike Jaklitsch for checking parts of the sixth chapter.
- Charles Klein for warm friendship and hot meals.
- Mark Madsen for lively discussions, his critical reading of the manuscript, his cynical sense of humor, and for friendship throughout.

My warmest thanks to you all.

I could't forget to say thank to the University of Cape Town and the Council for Scientific and Industrial Research for the financial support.

ABSTRACT

An investigation of three different cosmological topics is presented in this Thesis. In Chapters 2, 3 and 4 a review of spatially homogeneous cosmologies is presented setting up the mathematical preliminaries for the discussion of the dynamics of the Bianchi Universes contained in the following Chapters. The first investigation concentrates on the applicability of the automorphism group of the Bianchi Lie algebras in reducing the time dependence of the metric of such models. The simplified metric and the Einstein field equations are given for all non-abelian non-semisimple Bianchi types. A qualitative analysis of the vacuum, orthogonal and tilted perfect fluid field equations is also included. The second investigation is concerned with the nature of the initial singularity in the Friedmann-LeMaitre-Robertson-Walker Universes as seen by a fast-moving observer. Finally, an investigation of the smoothing-out of local inhomogeneities at different scales of description of the Universe is presented in the last Chapter of the Thesis. Each of these investigations is followed by some discussion and concluding remarks.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Outline

For the past thirty yeras or so the dynamics and geometry of the large scale structure of the Universe have been investigated mainly in the light of the Spatially Homogeneous Cosmological Models. The reason for that is at least two-fold: Firstly, these models represent a large class of 'physically reasonable' cosmological models, as they agree with the current observational status of homogeneity; and secondly, they are mathematically more tractable than the inhomogeneous models, as in this case the Einstein field equations involve only ordinary (non-linear) differential equations of a single variable (which can be chosen to be the cosmic time t) rather than involving partial differential equations in all space-time variables x^a.

A space-time is spatially homogeneous when there is a continuous group of isometries acting transitively on 3-dimensional spacelike hypersurfaces. These hypersurfaces are then 'surfaces of homogeneity' because any physical quantity has the same value everywhere on them. We shall denote the surfaces of homogeneity by S(t), where t is a parameter along the integral curves of the normal vector field to these surfaces. The group of isometries may act simply or multiply transitively according to the number of Killing vector fields in the surfaces S(t). The models with a multiply transitive action fall into two categories: models with a 6-dimensional group of isometries and models with a 4-dimensional group of isometries. The simply transitive models have a 3-dimensional group of isometries.

The Friedmann-LeMaître-Robertson-Walker (FLRW) models constitute the class of models with a 6-dimensional group of isometries. Three of the Killing vector fields are responsible for the spatial homogeneity and the other three Killing "not the" vector fields form a 3-dimensional subgroup of isotropies, which leave the orthogonal direction to S(t) invariant. Thus the FLRW Universes are spatially isotropic in addition to their spatial homogeneity. The models with a 4-dimensional group of isometries are called Locally Rotationally Symmetric (LRS) (Ellis, 1967) because they have a 1-dimensional isotropy subgroup. [The Kantowski-Sachs model has a 4-dimensional group of isometries, however there is no 3-dimensional subgroup that acts simply transitively on the 3-dimensional spacelike hypersurfaces (see Kantowski and Sachs, 1966 and also Collins, 1976)].

Strictly speaking, Bianchi Universes are the spatially homogeneous models which have a 3-dimensional group of isometries acting simply transitively on the surfaces of homogeneity. They have no continuous isotropy subgroups. However in general the FLRW and LRS (except the Kantowski-Sachs case) models are considered as being Bianchi models with additional symmetries as they always have a 3-dimensional subgroup that acts simply transitively on S(t) plus a 1- or 3-dimensional isotropy subgroup.

no non-trivial

The dynamics of the Bianchi Universes has been lengthily investigated by many authors (Taub, 1951; Heckmann and Schuking, 1962; Ellis and MacCallum, 1969; MacCallum, 1973; Ryan and Shepley, 1975; MacCallum, 1979b). However in all these investigations one of two approaches have been adopted in order to obtain the Einstein field equations. The first approach considers an orthonormal tetrad basis where the 3-spacelike vectors are the invariant vector fields under the 3-dimensional group of isometries. Thus the space-time time dependence remains in the structure functions and the Einstein equations determine the evolution of the expansion, shear and rotation of the normal congruence (cf. Ellis and MacCallum, 1969). The second approach is obtained when we choose a (non-orthonormal) tetrad basis where the 3-spacelike invariant vector fields are also chosen to be invariant by dragging along the normals to the surfaces S(t). Thus in this case the time dependence remains in the metric components and the structure functions become constant everywhere in the space-time (cf. Ryan and Shepley, 1975 and Heckmann and Shucking, 1962).

Recently in a short letter, Siklos (1980) has suggested that by combining the two approachs a relatively simpler form for the metric and field equations for Bianchi Universes can be obtained. The central idea relies on the use of the time-dependent automorphism group of the Bianchi Lie algebras to simplify the metric time dependence and hence the field equations. The automorphism group is the group of transformations that leave the Bianchi symmetry group structure constants invariant. In other words the canonical form of the structure constants is preserved under the automorphism group action.

As an application of this method to simplify the metric and field equations, Siklos considered the Bianchi type VI_h space-time and then briefly discussed extending his results to other class B Bianchi Universes. However the modifications pointed out by him to obtain such extension are shown in this thesis to be incorrect and incomplete. Additionally, Siklos made no further attempt to consider the application of this method to the class A Bianchi Universes. Here we also discuss the application of this method to the class A models with particular attention to Bianchi type II.

The application of this method has been previously considered by Collins and Hawking (1973) and also by Bogoyavlenski and Novikov (1973). However both papers deal with specific problems and particular group types; they did not consider the extension to general Bianchi types. The most extensive use of the automorphism group in this direction has been contained in the work done by Jantzen (1979, 1983), where the Hamiltoniam formalism and the scale invariance of the Einstein field equations were employed to investigate the dynamics of the perfect fluid spatially homogeneous cosmologies.

In Chapters 5 and 6 of this thesis we describe this method systematically and discuss the usefulness of the time dependent automorphism group of the Bianchi Lie algebras as a tool to simplify the metric and field equations for the Bianchi Universes. The topic will be presented in a new simple geometric fashion, rather than relying on large segments of Group Theory as in the Hamiltomain approach developed by Jantzen (1979, 1983). A number of outstanding points will be clarified from the Siklos paper, in particular the field equations are presented in a form covering all Bianchi types and the full automorphism group and invariance family for each Bianchi type is given in detail. The detailed investigation of the field equations is done using the Bianchi type VII_h as the foundation for the computations. However, as will be shown in section 6.3, a simple extension of the Bianchi type VII_h results allow some other Bianchi types to be easily considered. The Bianchi types V and II are discussed individually because they have a larger automorphism group which gives extra freedom to further simplify the metric. The results presented here form a coherent geometrically simple approach throwing new light on the field equations simplified by the use of the automorphism group.

In a recent paper Rosquist (1984) has presented the regularized field equations for the Bianchi type VI_h Universes as a 7-dimensional autonomous system. This system admits a 4-dimensional Taub subsystem which is sufficiently general to allow rotation of the matter flow lines. While investigating equilibrium solutions of the Taub subsystem, Rosquist has found an exact expanding and rotating solution with constant tilt among the class A Bianchi type VII_0 Universes filled with radiation (Rosquist, 1984 and 1983). A closer investigation of the Taub subsystem has been done by myself and M. Jaklitsh. We have found three other solutions for the Taub equilibrium subsystem corresponding to vacuum, dust and radiation Bianchi type VI_0 Universes. The dust and radiation solutions are non-tilted and belong to the family of solutions for Bianchi type VI_0 previously found by Collins (1971). However the vacuum solution has shown to be inconsistent with the results obtained in Chapter 6 of this thesis, namely, that for the class A vacuum Bianchi types VI_0 and VII_0 the special automorphism group parameters must be constant. This is in confrontation with the vacuum 'solution' found from the Taub subsystem given by Rosquist (1984). In fact, we have found that the vacuum case in the whole Hamiltonian formalism of Jantzen should be treated as a specific case, or at least with caution, as in this case the Hamiltonian and momentum constraints cannot be used to replace the matter variables for the geometric ones, which represents one of the major achievements in the Jantzen Hamiltonian formulation.

In a mixture of the Jantzen approach and the geometric approach developed here, we have shown that, in fact, the correct Taub subsystem for the vacuum Bianchi type VI_0 does not admit an equilibrium solution. A similar result also holds for the compactified autonomous system for the vacuum Bianchi type VII_0 Universe. The dust and radiation solutions, the correct Hamiltonian formulation for the vacuum case for the non-abelian non-semisimple class B Bianchi types, and a 1-parameter families of vacuum solutions for the class B Bianchi types IV, VI_h and VII_h will appear elsewhere (a preprint copy of this article is given in the Appendix).

It is a well established result that all spatially homogeneous Universe models with 'reasonable physical conditions' originate at initial singularity. This result is proved by the singularity theorems developed in the early seventies by Hawking and Penrose (Hawking and Penrose, 1970; see also Hawking and Ellis, 1973). Intuitively, a singularity is a 'place' where some quantities such as the energy density or the scalar curvature become unbounded; in other words they become infinite at the singularity. Many different kinds of space-time singularities have been found in the context of spatially homogeneous cosmologies. The most common singularities are the Big-Bang type of singularity, however a variety of different sorts of singularities occur in the Bianchi Universes, in particular for the tilted models (cf. Ellis and King, 1974; Ellis and Schmidt, 1977 and Collins and Ellis, 1979).

The singularity theorems guarantee the existence of space-time singularities, nevertheless they give very little information concerning the detailed nature of those singularities. As an example, they do not distinguish where the singularity occurs (whether in the future or in the past), or how strong the singularity is (see Tipler, 1977), or the behaviour of particles approaching to it. Thus, most of the current investigations on singularities have been concentrated essentially on the understanding of their nature rather than trying to find or prove more general statements on their existence. ??

Chapter 7 is dedicated to the problem of the nature of the singularity. There an investigation of the nature of the initial singularity in FLRW Universes is carried out. It is shown that different observers have different views of the singularity. In particular, for a fast moving observer approaching the singularity, the rate of change of any spatially homogeneous quantity measured by him will be slower than that measured by a fundamental observer. However, this effect is cancelled out when the measurements are performed in the proper time of each observer. The fast moving observer experiences much stronger gravitational tidal forces compressing him in the direction of the motion than in the perpendicular directions. Thus the fast moving observer is anisotropically compressed as he falls into the singularity in contrast to the isotropic ??

compression experienced by the fundamental observer. These effects might not have major implications for the internal structure of particles or their interactions near the singularity, but certainly they change our view of the isotropic singularity and hence of its nature - because most particles in the early Universe will indeed be moving fast relative to a fundamental observer.

So far our 'best-fit' Universe models have assumed that at large scale (that is > 100 Mpc.) the Universe looks spatially homogeneous and isotropic. However, this is not true when smaller scales (< 100 Mpc.) are taken into account. In fact, at the scale where the stars and galaxies are seen as individual objects, the Universe looks highly inhomogeneous and anisotropic. Thus it would be useful to have cosmological models considering the Universe at different scales of description, where the ultimate scale would be in good agreement with the present observational status of spatial homogeneity and isotropy at large; in other words, to have a cosmological model which at small scales describes the local inhomogeneities but through a continuous 'smoothing-out' procedure approaches at large scales to the FLRW Universes.

One way to construct these 'Lumpy Universes Models' is by defining different scales of description of the Universe, where the large scale description is obtained by smoothing-out that at the small scale. The idea is thus to have a series of metrics that describe the space-time properties at different scales, the transition from one to another being a result essentially of the smoothing-out of the metric on each scale. This immediately suggests defining 'smoothing-out operators' ~~such~~ that when applied to a small scale quantity ~~it~~ give~~s~~ the corresponding smoothed-out one. These smoothing-out operators should, of course, satisfy certain conditions as for example composition.

[stet]

~~as to~~

How to define a smoothing-out operator is unclear, ~~and to~~ whether to apply it first to the small scale metric or to some other small scale quantity (such as cit.) with the 1-parameter families of vacuum solutions for Bianchi types IV, VI_h and VII_h.

1.2 Notation and Conventions

In this thesis we assume:

- Latin letters a, b, c... are used to refer to a general basis while the letters i, j, k... are used to refer to a coordinate basis. Greek letters α, β, γ... are used to represent the spatial part of a geometric quantity.
- The symbol $\otimes$ means tensor product.
- If M is a Riemannian manifold, T_pM stands for the tangent space of M at p, where p is a point of M. T^*_pM denotes the cotangent space or dual space to the tangent space. TM and T^*M represents the tangent and cotangent bundles respectively.
- Any vector field $V \in TM$ will be written in terms of a general vector basis $\{E_a\}$ as

$$V = v^a E_a = v^a E_a{}^i \partial_i,$$

where $\{\partial_i = \partial/\partial x^i\}$ denotes a coordinate basis. The dual to $\{E_a\}$ is denoted by $\{E^a\}$, where $\langle E^a, E_b\rangle = \delta^a{}_b$ defines the duality. Any 1-form field will be written in terms of $\{E^a\}$ as

$$\omega = \omega_a E^a = \omega_a E^a{}_i dx^i,$$

with $\{dx^i\}$ the dual basis of $\{\partial_i\}$.

the energy density or the shear) in order to obtain the smoothed-out metric describing the large scale properties of the space-time, is open to question. The last chapter of this thesis is dedicated to the investigation of such ideas. We give a general picture of the properties of the smoothing-out operators and discuss how the Einstein field equations change when applied to different scales. A particular smoothing-out operator is suggested and the energy momentum tensor describing the large scale properties of the space-time is given. There is no evidence the strong energy condition for the smoothed-out scale is satisfied for all observers, even if it is assumed to be valid for the small scale. However, Carfora and Marzuolli (1984) claim that in the transition from locally inhomogeneous and anisotropic spatially closed Universes into closed FLRW Universes ($\Lambda = 0$) the dominant energy condition is not violated. On the other hand, in the Turbulance papers of Marochnik et al. (1975), where an averaging operator has also been defined, the turbulance energy and pressure can both be negative, and enter into the field equations describing the system in an equal footing with the energy and presssure of matter. Actually, they have concluded that the singularity in an Universe which is homogeneous and isotropic on the average, is not inevitable. Thus it is unclear if the singularity theorems apply to the smoothed-out (large scale) scale of description, assuming they do at the small scale.

In summary, the Thesis is divided as follows: i) Chapters 2, 3 and 4 give an introduction to spatially homogeneous cosmologies and establish the mathematical preliminaries, ii) Chapters 5 and 6 are concerned with the automorphism group of the Bianchi Lie algebras and their application to simplify the metric of Bianchi Universes, iii) in Chapter 7 an investigation of the nature of the initial singularity in FLRW Universes seen by a fast moving observer is presented and finally, iv) Chapter 8 is dedicated to the smoothing-out problem of different scales of the Universe. In the Appendix we reproduce a copy the preprint (op.

- A vector field V operating on a geometrical quantity, f say, is denoted by

$$V(f) = v^a E_a(f) = v^a \partial_a f.$$

- Symmetrization is denoted by round brackts '( )' and antisymmetrization by square-brackets '[ ]'.
- The sign convention for the curvature tensor is defined in equation (2.35).
- The system of units is chosen such that $8\pi G = c = 1$, unless stated otherwise.

CHAPTER 2

RIEMANNIAN MANIFOLDS AND SPACE-TIME STRUCTURE

The aim of this chapter is to give a summary of the basic concepts of a Riemannian Manifold and to describe the Space-Time structure. It is assumed that the elementary notions of differential geometry are known. Many references in which further details may be found are given in the text.

2.1 Riemannian Manifolds

A smooth[1] manifold M is a <u>Riemannian Manifold</u> if a continuous tensor field g of type (0,2), called the <u>metric tensor</u>, is defined on it such that g satisfies:

1) $g(V,U) = g(U,V)$, for all V,U on TM.

2) $g(V+V',U) = g(V,U) + g(V',U)$,

3) $g(aV,U) = ag(V,U)$, $a \in R$,

4) $g(V,U) = 0$ for all V iff $U = 0$,

5) $g(V,V) > 0$.

If property 5) does not hold the manifold is called a <u>Pseudo-Riemannian Manifold</u> (Choquet-Bruhart et al.,1978). We will be dealing with pseudo-Riemannian manifolds.

The components of the metric tensor g with respect to a general vector basis $\mathbf{E}_a$ are given by (Hawking and Ellis,1973)

$$g_{ab} = g(\mathbf{E}_a,\mathbf{E}_b) = \mathbf{E}_a \cdot \mathbf{E}_b, \tag{2.1}$$

which is simply the Euclidean scalar (or dot) product of the vector basis $\mathbf{E}_a$. In a coordinate basis we have

$$g = g_{ij}\, dx^i \otimes dx^j; \tag{2.2}$$

as g_{ij} is symmetric, the relation (2.2) is normally written as

$$\mathbf{g} = ds^2 = g_{ij}\, dx^i dx^j. \tag{2.3}$$

The property 4) enables us to define a unique symmetric tensor of type (2.0) having components g^{ab} with respect to a general vector basis $\{\mathbf{E}^a\}$ dual to $\{\mathbf{E}_a\}$ by the relations

$$g^{ac} g_{cb} = \delta^a{}_b. \tag{2.4}$$

In other words the matrix (g^{ab}) is just the inverse matrix to (g_{ab}). An isomorphism can then be set up between contravariant and covariant tensors via raising and lowering of indices by use of g^{ab} and g_{ab}, respectively (Choquet-Bruhat et al.,1978). Let v^a be the contravariant components of a vector field V; then v_a are the covariant components of V uniquely associated to v^a by

$$v_a = g_{ab}\, v^b \iff v^a = g^{ab}\, v_b. \tag{2.5}$$

Equations (2.5) are extended to tensors similarly.

The signature of the metric tensor g is defined as the difference between the number of its positive and negative eigenvalues at a point p of M. When the signature of g is sign(g) = n - 2, with n the dimension of the manifold, we say that g is a Lorentzian metric (Hawking and Ellis,1973). For the case of n = 4, with which we will be mainly concerned, the manifold is called hyperbolic, and its metric is said to be a hyperbolic metric.

A null cone structure may be defined on a hyperbolic manifold by

$$g(V,V) = 0,\ V \neq 0 \iff g_{ab}v^a v^b = 0. \qquad (2.6)$$

A vector field that satisfies equation (2.6) is called null. If V is a vector field such that

$$g(V,V) < 0 \iff g_{ab}v^a v^b < 0, \qquad (2.7)$$

$$g(V,V) > 0 \iff g_{ab}v^a v^b > 0, \qquad (2.8)$$

then we say that V is a timelike or spacelike, respectively.

A connection[2] ∇ at a point p of M is a map which assigns to each vector field V at p a differential operator ∇_V which when applied to a C^k-vector field U gives a C^{k-1}-vector field $\nabla_V U$, with the following properties:

1) $\nabla_{V+U}W = \nabla_V W + \nabla_U W$ (2.9)

2) $\nabla_V(U+W) = \nabla_V U + \nabla_V W$ (2.10)

3) $\nabla_V f = V(f)$ (2.11)

4) $\nabla_{fV}U = f\nabla_V U$ (2.12)

5) $\nabla_V(fU) = f\nabla_V U + V(f)U,$ (2.13)

where V, U, and W are vector fields and f is a scalar function.

The covariant derivative of a vector field U with respect to V at a point p of M is defined as $\nabla_V U$. If $\{E_a\}$ and $\{E^a\}$ are vector field and 1-form field bases, then the covariant derivative of U in the direction of V is given by (Hawking and Ellis,1973)

$$\nabla_V U = (u^a{}_{;b}v^b)E_a, \qquad (2.14)$$

where

$$u^a{}_{;b} = u^a{}_{,b} + \Gamma^a{}_{bc}u^c, \qquad (2.15)$$

and since $\langle E^a, E_b\rangle = \delta^a{}_b$,

$$\nabla_{E_b} E_a = \Gamma^c{}_{ba} E_c \quad \Longleftrightarrow \quad \Gamma^c{}_{ba} = \langle E^c, \nabla_{E_b} E_a \rangle \tag{2.16}$$

The n^3 terms $\Gamma^c{}_{ab}$ defined in equation (2.16) are the <u>connection coefficients</u> and relate the basis at different points of M. The covariant derivative is extended to tensors in similar manner (Hawking and Ellis,1973). should define using (x)

A vector field U is <u>parallel transported</u> along a curve $\gamma(t)$ if (Hicks,1965 and Choquet-Bruhat et al.,1978)

$$\nabla_V U = 0, \tag{2.17}$$

where the vector V is the tangent vector to the curve $\gamma(t)$. The vector $\nabla_V U$ is sometimes denoted by DU/Dt. A curve $\gamma(t)$ is a <u>geodesic</u> iff

$$\nabla_V V = f(t)V, \tag{2.18}$$

where f is a scalar function. When f=0, the parameter t is called an <u>affine parameter</u>. Parallel transport can also be extended to tensors (see Hawking and Ellis,1973).

2.2 The Torsion Tensor

The <u>torsion tensor</u> is defined as

$$S(V,U) = \nabla_V U - \nabla_U V - [V,U], \tag{2.19}$$

where V and U are arbitrary vector fields and [V,U] is the <u>commutator</u> of V and U defined by

$$[V,U]f = V(U(f)) - U(V(f)), \tag{2.20}$$

with f a scalar function.

In a general basis $\{E_a\}$ the torsion tensor components are given by

$$S^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb} - \gamma^a{}_{bc}, \tag{2.21}$$

where the $\gamma^a{}_{bc}$ are the defined from the vector basis commutators by

$$[E_b, E_c] = \gamma^a{}_{bc} E_a. \tag{2.22}$$

The commutators defined in equation(2.22) are such that

$$[E_b, E_c] = -[E_c, E_b] \quad \Longleftrightarrow \quad \gamma^a{}_{bc} = -\gamma^a{}_{cb} \tag{2.23}$$

$$[E_a,[E_b,E_c]] + [E_c,[E_a,E_b]] + [E_b,[E_c,E_a]] = 0 \Longleftrightarrow$$

$$\partial_{[a}\gamma^d{}_{bc]} + \gamma^e{}_{[bc}\gamma^d{}_{a]e} = 0, \tag{2.24}$$

where equation (2.24) are the <u>Jacobi identities</u>.

The set $\{E_a\}$ is called a <u>coordinate basis</u> or <u>holonomic basis</u> (Schouten, 1954) iff all the commutators vanish, i.e.,

$$[E_a, E_b] = 0.$$

In a coordinate basis the torsion tensor is given by

$$S^i{}_{jk} = \Gamma^i{}_{jk} - \Gamma^i{}_{kj}. \tag{2.25}$$

When $S^a{}_{bc} = 0$, equation (2.21) reduces to

$$\gamma^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb}, \tag{2.26}$$

thus, in the particular case of a coordinate basis we have

$$\Gamma^i{}_{jk} = \Gamma^i{}_{kj}, \tag{2.27}$$

that is, the connection is symmetric. These are just the usual <u>Christoffel symbols</u>.

actually Christoffel symbols are if $\nabla_x g = 0$.

In a Riemannian manifold M there exists a unique linear connection such that

1) $S^a{}_{bc} = 0$, i. e., the torsion tensor vanishes

2) $g_{ab;c} = 0$, i. e., the covariant derivative of the metric tensor vanishes.

(See Choquet-Bruhat et al., 1978 for proof). The connection that satifies 1) and 2) is called a Riemannian connection.

The Riemannian connection coefficients are given, in a general basis $\{E_a\}$, by

$$\Gamma_{abc} = 1/2\,(g_{ab,c} + g_{ca,b} - g_{bc,a} + \gamma_{abc} + \gamma_{cab} - \gamma_{bca}), \tag{2.28}$$

where $\Gamma_{abc} = g_{ad}\Gamma^d{}_{bc}$. In a coordinate basis they are

$$\Gamma^i{}_{jk} = 1/2\, g^{il}(g_{lj,k} + g_{kl,j} - g_{jk,l}), \tag{2.29}$$

which is the well known definition of the Christoffel symbols in terms of the metric.

We will assume from now on that the torsion tensor vanishes, $S^a{}_{bc} = 0$, and the connection satisfies $g_{ab;c} = 0$.

2.3 The Curvature Tensor

It is a well know result that the covariant derivatives do not in general commute. This non-commutativity is in fact due to the presence of curvature of the manifold. The curvature tensor R is defined by

$$R(V,U)W = \nabla_V(\nabla_U W) - \nabla_U(\nabla_V W) - \nabla_{[V,U]}W, \tag{2.30}$$

where V,U and W are vector fields.

In a general basis $\{E_a\}$ the <u>curvature</u> or <u>Riemann tensor</u>, components are given by

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \Gamma^a{}_{cf}\Gamma^f{}_{db} - \Gamma^a{}_{df}\Gamma^f{}_{cb} + \Gamma^a{}_{fb}\gamma^f{}_{dc}. \tag{2.31}$$

The curvature tensor has the following properties:

$$R_{abcd} = -R_{abdc} = -R_{bacd}, \tag{2.32}$$

name [illegible]?

$$R^a{}_{[bcd]} = 0, \text{ \underline{Jacobi identities}}, \tag{2.33}$$

$$R^a{}_{b[cd;f]} = 0, \text{ \underline{Bianchi identities}}. \tag{2.34}$$

Any non-zero vector field V in M satisfy ies the <u>Ricci identity</u> (which is equivalent to eq. 2.30) in a coordinate basis.

$$v^a{}_{;bc} - v^a{}_{;cb} = -R^a{}_{dbc}v^d. \tag{2.35}$$

The <u>Ricci tensor</u> is defined by the contraction of the Riemann tensor as

$$R_{ab} = R^c{}_{acb}. \tag{2.36}$$

It is a symmetric tensor, $R_{ab} = R_{ba}$. The Ricci tensor components are given by

$$R_{ab} = \partial_c \Gamma^c{}_{ba} - \partial_b \Gamma^c{}_{ca} + \Gamma^c{}_{cd}\Gamma^d{}_{ba} - \Gamma^c{}_{da}\Gamma^d{}_{cb}. \tag{2.37}$$

A Riemannian manifold is said to be <u>flat</u> when the curvature tensor R vanishes at all points p of M. When the Ricci tensor vanishes ($R_{ab} = 0$) everywhere in M, the manifold is said to be <u>Ricci flat</u>.

The <u>curvature</u> or <u>Ricci scalar</u> is defined by

$$R = R^a{}_a = g^{ab}R_{ab}. \tag{2.38}$$

By contraction of the Bianchi identities (2.34) we get

$$R_{ab;c} - R_{ac;b} + R^{d}{}_{abc;d} = 0, \tag{2.39}$$

which (multipling by g^{ac}) gives

$$(R^{a}{}_{b} - 1/2\, R\delta^{a}{}_{b})_{;a} = 0. \tag{2.40}$$

Equations (2.40) are known as the <u>contracted Bianchi identities</u>.

2.4 The Space-Time Structure of General Relativity

Space-time structure is the primary concept of the Theory of General Relativity. Since here we are interested in the geometrical and dynamical aspects of a class of relativistic cosmological models, it is relevant to give the definition of space-time and some of its characteristics.

Let (M,g) and (M',g') represent two Riemannian manifold structures. They are said to be <u>isomorphic</u> when there exists a diffeomorphism $\phi : M \to M'$ such that its induced map ϕ_* carries g into g', i.e., $\phi_* g = g'$. The diffeomorfism ϕ is called an isomorphism (Boothby, 1975).

not defined.

A <u>space-time</u> is the class all isometric pairs (M,g) where M is a 4-dimensional smooth Riemannian manifold and g is a hyperbolic metric with signature +2 on M (Hawking and Ellis, 1973).

The <u>Einstein tensor</u> is a symmetric second order tensor G_{ab} defined by

$$G_{ab} = R_{ab} - 1/2\, Rg_{ab}. \tag{2.41}$$

From equation (2.40) we see immediately that the Einstein tensor is divergenceless,

$$G^{ab}{}_{;b} = 0, \tag{2.42}$$

as a result of the contracted Bianchi identities.

In a space-time there might be "matter fields" such as scalar or eletromagnetic fields, comprising the matter content of the space-time. The matter fields are discribed by a second order tensor, denoted T_{ab}, called the energy-momentum tensor field, such that it obeys the following postulates (Hawking and Ellis, 1973) stress-energy tensor

1) Local causality, which states that no signal joining any two points of the space-time can travel with a speed greater than that of light.

2) Local conservation of energy, which states that the energy-momentum tensor of the matter fields must be conserved, that is,

$$T^{ab}{}_{;b} = 0. \tag{2.43}$$

3) The Einstein Field Equations,

$$G_{ab} = T_{ab}, \tag{2.44}$$

must hold on the space-time. (Throughout this thesis we use the system of units with c = 8πG = 1, Sachs and Wu, 1977).

Sometimes the Einstein field equations are written in the more general form

$$R_{ab} - 1/2\, Rg_{ab} + \Lambda g_{ab} = T_{ab}, \tag{2.45}$$

where Λ is the cosmological constant. Equation (2.45) is divergence free

$$(R^{ab} - 1/2\, Rg^{ab} + \Lambda g^{ab})_{;b} = T^{ab}{}_{;b} = 0. \tag{2.46}$$

The equations (2.45) may equivalently be written as

$$R_{ab} = T_{ab} - 1/2\, Tg_{ab} + \Lambda g_{ab}, \tag{2.47}$$

where $T = T^a{}_a$ is the trace of the energy momentum tensor. stress-energy

The energy momentum tensor for each matter field present in the space-time can be found from the appropriate Lagrangian (Hawking and Ellis, 1973). We will not concern ourselves here with the construction of the energy momentum tensor from the Lagrangian. For details the reader should consult Carmeli, 1983 and Birrell and Davies, 1982.

The energy momentum tensor in which we are particularly interested here is that for a <u>perfect fluid</u> which is (Ellis, 1971) heuristic form, not necessarily from a Lagrangian.

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab}, \tag{2.48}$$

where μ is the energy density of the field, p is the isotropic pressure and u_a is the fluid 4-velocity, chosen to satisfy $u_a u^a = -1$. The pressure p is assumed to be related to the energy density μ via an <u>equation of state</u>

$$p = p(\mu), \tag{2.49}$$

which we shall choose to be of the form

$$p = (\gamma - 1)\mu, \quad \gamma = \text{constant}, \tag{2.50}$$

where $1 \leq \gamma \leq 2$. The case $\gamma = 1 \iff p = 0$ is called <u>dust</u>; the case $\gamma = 4/3 \iff p = (1/3)\mu$ is called <u>radiation</u> and the extreme case of $\mu = 2 \iff p = \mu$ is called <u>stiff matter</u>.

Footnotes:

1) By smooth we mean paracompact connected C^∞ Hausdorff manifold without boundary (see Hawking and Ellis, 1973).

2) See von Westenholz, 1978, Chapter 11, for some definitions of connections.

CHAPTER 3

CONTINUOUS GROUPS OF TRANSFORMATIONS

The Theory of Lie Groups and Lie Algebras plays an important role in the description of the continuous symmetries of the space-time. As we will see in Chapter 4, the continuous symmetries of a space-time are given in terms of Killing vector fields which are the generators of the Lie algebra of the Lie group of isometries. The existence of at least three non-co-planar Killing vector fields acting on space-like hypersurfaces reduces the Einstein field equations from non-linear partial differential equations to non-linear ordinary differential equations in a single variable, which can be choosen to be the cosmic time t.

In this Chapter an introduction to continuous Lie groups of transformations is given. As in Chapter 2, we will concern ourselves mainly with the nomenclature and results, rather than with their development or proof, for which the reader may consult references cited in the text.

3.1 Lie Groups and Lie Algebras

In this section, the basic notions of the theory of abstract groups are assumed to be known. There are many books on the subject; the reader may consult the excellent introductory work of Hamermesh, 1962, or the review article of Gursey, 1962.

A Lie Group is a set which has at the same time both a group structure and a manifold structure, and where the group operation

$$G \times G \to G$$
$$(a,b) \to ab^{-1},$$

is a C^∞-map (Cohn, 1957; Sagle and Walde, 1973).

A trivial Lie group is the n-dimensional real manifold R^n under the operation of (vector) addition. One of the most important Lie groups is the General Linear Group in n real dimensions $GL(n,R)$, which is the set of all non-singular nxn real matrices under the operation of matrix multiplication (von Westenholz, 1978).

A natural map defined on a Lie group G is a Left Translation

$$L_a : G \to G$$
$$b \to L_a b = ab \tag{3.1}$$

Similarly we can define a Right Translation

$$R_a : G \to G$$
$$b \to R_a b = ba. \tag{3.2}$$

Both the left and right translations are diffeomorphisms, and commute (Cohn, 1957 and MacCallum, 1973).

A left (right) translation L_a (R_a) induces a map $(L_a)_*$ ($(R_a)_*$) from the tangent space of G at $b \in G$, $T_b(G)$, to the tangent space $T_{ab}(G)$, for all a of G (von Westenholz, 1978). This enables us to define left (right) invariant vector fields as follows:

A vector field V on G is said to be left (right) invariant vector field if for every a, $b \in G$ we have

$$(L_a)_* V(b) = V(L_a b) = V(ab), \tag{3.3}$$

$$(R_a)_* V(b) = V(R_a b) = V(ab), \tag{3.4}$$

respectively. From (3.3) and (3.4) we see immediately that

$$(L_a)_* V(e) = V(a), \text{ all } a \in G, \tag{3.5}$$

$$(R_a)_* V(e) = V(a), \text{ all } a \in G, \tag{3.6}$$

where e is the identity element of G, such that ae = ea = a.

The equations (3.5) and (3.6) state that the value of V can be known everywhere on G as long as we known its value at the identity e of G and the action of the left or right translations. Therefore a left (right) invariant vector field is defined globally on G.

3.2 Lie Algebras

A real Lie Algebra $\underline{G}$ is a finite dimensional real vector space such that the product of two vectors, defined as [X,Y], satisfies

1) bilinearity over R

$$[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z]$$

2) antisymmetry

$$[X,Y] = - [Y,X]$$

3) Jacobi identity

$$[X,[Y,Z]] + [Z,[Y,X]] + [Y,[Z,X]] = 0,$$

where α and $\beta \in R$, and X, Y and Z are vectors in $\underline{G}$.

Let $\{X_a, a = 1...r\}$ be a basis for a Lie algebra $\underline{G}$. As the commutator of two vector fields is a vector field (Choquet-Bruhat et. al., 1978), there must exist constants $C^a{}_{bc}$ such that

not the reason: here [,] is a Lie algebra product; later identify [,] with vector field commutation

$$[X_b, X_c] = C^a{}_{bc}X_a, \quad X_a \in \underline{G}. \tag{3.7}$$

The constants C's are the <u>structure constants</u> of the Lie algebra $\underline{G}$. It follows from 1), 2) and 3) that the structure constants transform under a change of basis of $\underline{G}$ as a third order tensor and that

$$C^a{}_{bc} = - C^a{}_{cb}, \tag{3.8}$$

$$C^f{}_{[bc}C^a{}_{d]f} = 0, \tag{3.9}$$

the last is a result of the Jacobi identity.

The vector subspace formed by the commutators

$$D = \{[X,Y] \; ; \; X \text{ and } Y \in \underline{G}\}, \tag{3.10}$$

forms the <u>derived algebra</u> of $\underline{G}$.

We say that a Lie algebra is <u>abelian</u> when

$$[X,Y] = 0, \text{ for all } X, Y \in \underline{G};$$

in other words if its derived algebra is 0-dimensional.

The commutator of two left (right) invariant vector fields is also a left (right) invariant vector field; therefore the set of all left (right) invariant vector fields generates a Lie algebra. We define the <u>Lie algebra of a Lie group</u> G as the Lie algebra of its left invariant vector fields. It can be shown that the dimension of the Lie algebra $\underline{G}$ is equal to the dimension of the Lie group G (von Westenholz, 1978).

The right invariant vector fields form a Lie algebra such that if $\{\underline{\Delta}_a, a =$

$1 \ldots r\}$ is a basis for the algebra $\underline{G}$ of the right invariant vector fields then (MacCallum, 1979a)

$$[X_a, \underline{X}_b] = 0. \qquad (3.11)$$

If we define the structure constants of $\underline{G}$ by

$$[\underline{X}_a, \underline{X}_b] = D^c{}_{ab}\underline{X}_c, \qquad (3.12)$$

then since $\{X_a\}$ and $\{\underline{X}_a\}$ are bases of T(G) we have that

$$\underline{X}_a = \psi_a{}^b X_b. \qquad (3.13)$$

From equations (3.12) and (3.13) we can show that by an appropriate choice of $\psi_a{}^b = -\delta_a{}^b$, we have

$$D^a{}_{bc} = - C^a{}_{bc}, \qquad (3.14)$$

which shows that the Lie algebras are isomorphic. Any Lie group has a unique Lie algebra associated to it. However the converse is only true up to homomorphism (Cohn, 1957).

3.3 Lie Groups of Transformations

Let M be an n-dimensional Riemanniam manifold and G an r-dimensional Lie group. We say that G acts on M if there is a map

$$T : G \times M \to M$$

such that T satisfies

1) $T(e,x) = x$, for all x in M and e identity of G,

2) $T(a,T(b,x)) = T(ab,x)$, for all x in M and a,b in G.

For the sake of simplicity the map $T : G \times M \to M$ is shortened to

$$\begin{aligned} T_a &: M \to M \\ & x \to T_a x = ax, \end{aligned} \tag{3.15}$$

where a is any element of G.

The set of all transformations T_a, $a \in G$, are such that: $T_a . T_b = T_{ab}$; $T_a . T_{a^{-1}} = T_e \Rightarrow T_{a^{-1}} = (T_a)^{-1}$, and thus form a group under composition. This group of transformations on M is called the <u>Lie group of transformations</u> of M (Eisenhart, 1962).

A group of transformations is said to act <u>effectively</u> on M when for every $x \in M$,

choice of word not clear here.

$$T_a x = x \Rightarrow a = e \in G.$$

When a Lie group G is r-dimensional, the associated group of transformations is said to have r-parameters. If the group of transformations acts effectively, then we say that it has <u>r-essential parameters</u>. From now on we shall assume that the group of transformations acts effectively, i.e., it has only essential parameters. We will denote this group by G_r.

The set of all points of M that can be reached from a single point of x of M by the action of the group of transformations G_r is called the <u>orbit</u> or <u>trajectory</u> of G_r at x. In other words the orbit of $x \in M$ is given by

$$O = \{ T_a x ; a \in G \} \tag{3.16}$$

The orbit is a submanifold of M. Of course the orbit dimension is always less than or equal to the group dimension r, i.e., $d \le r$ for d the orbit dimension.

even when the group acts effectively, as you've assumed

A submanifold M' of M is an <u>invariant manifold</u> of G_r when

$$T_a M' = M', \text{ for all } a \in G. \tag{3.17}$$

It is clear that the orbits of M are invariant submanifolds of M.

A group of transformations is said to act <u>transitively</u> on M if for any two points x and y of M there is at least one transformation $T_a \in G_r$ such that

$$T_a x = y.$$

The action is <u>simply transitive</u> when the transformation T_a that joins the two points x and y is unique. In other words, we say that the group G_r is simply transitive when there is one and only one way to go from a point x in M to any other point y in M by a single group action. This is equivalent to saying that $r = n$, where r is the group dimension. When $r > n$ the group G_r is said to act <u>multiply transitively</u>.

Although a group of transformations acts transitively on its orbits, this action can be simply transitive in some orbits and multiply transitive on others.

Then we define a subgroup of G_r such that it leaves a point x of M fixed. This subgroup is called the <u>stability group</u> of M at x and is defined by

isotropy group?

$$S_x = \{ T_a, a \in G; T_a x = x \}. \tag{3.18}$$

The stability group S_x at x of M is an invariant subgroup of the group of transformations G_r (Eisenhart, 1962).

The dimension of the stability group S_x at x is given by

$$s = r - d, \tag{3.19}$$

where d is the orbit dimension at $x \in M$ and r is the dimension of the group of isometries (Eisenhart, 1962; MacCallum, 1979a).

A manifold M is said to be a <u>homogeneous</u> <u>space</u> of a Lie group of transformations when the action of G_r on M is a C^∞-transitive action (Helgason, 1978).

The vector fields $\{\xi_a,\ a = 1...r\}$, which are the generators of a Lie group of transformations G_r, are called the <u>infinitesimal</u> <u>transformations</u> of M. They are linearly independent vector fields and form a Lie algebra isomorphic to the Lie algebra of G. Actually it can be shown that an infinitesimal transformation is a right (left) translation if and only if it is a left (right) invariant vector field (Cohn, 1957). Each infinitesimal transformation gives an one-parameter subgroup of $\underline{G_r}$.

When the group of transformations G_r acts simply transitively on its orbits we can find a set of linearly independent vector fields $\{W_a,\ a = 1...r\}$ spanning the tangent space of the orbit at $x \in M$ which commute with the generators of G_r, $\{\xi_a\}$,

$$[\xi_a, W_b] = 0.$$

The vector fields $\{W_a\}$ form a Lie algebra which is algebraically identical to the Lie algebra of G_r; thus they are the generators of another group of transformations called the <u>reciprocal</u> <u>group</u> of G_r (Eisenhart, 1962; MacCallum, 1979a).

3.4 Groups of Isometries

In section 2.4 of Chapter 2, we introduced the notion of isometric space-times when there is an isomorphism $\phi : M \to M'$ such that $\phi_* : g \to g'$. When $M \equiv M'$

we can use the definition (3.16) to determine those transformations which are isometries, i.e., those transformations under which the metric tensor g is invariant.

A transformation T is called an *isometry* or *motion* when we have $(T)_* g = g$; i.e., when the metric is invariant under its action. Consequently all the space-time properties are also preserved. It can be shown that a transformation is an isometry if and only if its generator ξ satisf~~x~~ies

$$L_\xi \, g = 0, \tag{3.20}$$

where L is the Lie derivative (see Hawking and Ellis, 1973 for the definition of the Lie derivative). Equation (3.20) is equivalent to

$$\xi_{i;j} + \xi_{j;i} = 0, \tag{3.21}$$

which are known as *Killing's equations*. The vector fields which are the solutions to Killing's equations are called *Killing vector fields*; they are the generators of the continuos space-time isometries. The set of all Killing vector fields form a Lie algebra which is the Lie algebra of the Lie group of isometries.

The stability group S_x of a group of isometries is often called the *isotropy group* of M at x. This group is characterized by the vanishing of its generators at the point x. In what follows three theorems are given on the dimension of a group of isometries acting on a Manifold M. They are important to the classification of the Lie algebra of group of isometries:

Theorem 1: An n-dimensional Riemannian Manifold cannot have more than $\frac{1}{2}n(n + 1)$ linearly independent Killing vector fields.

Proof: (see MacCallum, 1979a, b.)

<u>Theorem</u> 2: An n-dimensional Riemannian Manifold does not admit a group of isometries with dimension $r = \frac{1}{2}n(n + 1) - 1$, if $n \geq 3$.

Proof: (see Eisenhart, 1962)

<u>Theorem</u> 3: An n-dimensional Riemannian Manifold has constant curvature if and only if it admits a group of isometries with dimension $r = \frac{1}{2}n(n+1)$; that is, with the maximal number of Killing vector fields.

Proof: (see Eisenhart, 1962 and MacCallum, 1979a.)

CHAPTER 4

SPATIALLY HOMOGENEOUS SPACE-TIMES

4.1 Spatial Homogeneity

A space-time is <u>homogeneous</u> when there is a group of isometries which acts transitively on the whole space-time manifold. Homogeneous space-times are of little interest in cosmology because they do not show any overall evolution. None of their physical quantities evolve with time; they are 'steady state' universes.

A space-time is <u>spatially homogeneous</u> when there is a group of isometries acting transitively on space-like 3-dimensional hypersurfaces (orbits). The 3-dimensional space-like hypersurfaces are the <u>surfaces of homogeneity</u> of the space-time. When there exists a Killing vector field which is everywhere time-like, the space-time is <u>stationary</u>; if in addition it is hypersurface normal the space-time is <u>static</u> (see Carmeli, 1982 and MacCallum, 1973). irrelevant

In a spatially-homogeneous space-time we have a G_r of isometries acting transitively on 3-dimensional orbits; therefore from theorem 1 on p. 19, we can only have $3 \leq r \leq 6$, since the orbit dimension is $d = 3$. Nevertheless from theorem 2 on p. 19, r can only assume the values: $r = 3$, 4 or 6.

When $r = 6$, from eq.(3.19) we have that $d = s = 3$. In this case there is a continuous group G_3 of isotropy at each point of the orbit. We have then the maximal number of linearly independent Killing vector fields acting on the spatially-homogeneous hypersurfaces which, from the result of theorem 3 on p.

20, leads to hypersurfaces of constant curvature. Those Universe models in which a G_6 of isometries acts transitively on space-like sections are known as <u>Friedmann-Robertson-Walker Universes</u>.

If $r = 4$, then we have $d = 3$ and $s = 1$. In this case there is a 1-dimensional continuous group of isotropy at each point of the space-like orbits; in this case the space-time is said to be <u>locally rotationally symmetric</u> (LRS) (Ellis, 1967; Ellis and Stewart, 1968).

For the case of $r = 3$, we have $d = 3$ and $s = 0$. The group of isometries G_3 acts simply-transitively on the space-like surfaces of homogeneity. There is no continuous isotropy group. Those space-times with a G_3 of isometries acting simply-transitively on the surfaces of homogeneity are called <u>Bianchi Universes</u>.

the term applies whenever a G_3 acts simply transitively, even if the isotropy group is nonzero.

For any G_6 of isometries that acts transitively on the surfaces of homogeneity there is a subgroup G_3 of isometries that acts simply-transitively on these surfaces (MacCallum, 1973). In the case of a G_4 of isometries we can always find a G_3 that acts simply-transitively on the surfaces of homogeneity with one particular exception: the Kantowski-Sachs Universe (Kantowski and Sachs, 1966; see also Collins, 1977). In general the Universe models with $r = 6$ or 4 (with the Kantowski-Sachs exception) are considered as being Bianchi Universes with additional symmetries.

A space-time can have at most 10 linearly independent Killing vector fields, i. e., a G_{10} of isometries. Such space-times are of constant curvature and conformal to Minkowiski space-time. G_{10} corresponds to the Poincare group which consists of 4 translations, 3 rotations and 3 boosts. From eq.(3.19) we immediately see that the maximal isotropy group of a space-time is a G_6, which is just the Lorentz group.

4.2 Spatially Homogeneous Metrics

In a spatially homogeneous space-time all points that lie on the surface of homogeneity are equivalent in the sense that there is an isometry joining them. Let us consider a vector n, normal to the surface of homogeneity at a point p. By dragging along this vector with the group of isometries we have n defined everywhere normal to the surface of homogeneity. If we now carry all points p of the surface of homogeneity along the integral curves generated by n we generate a family of surfaces of homogeneity, which we denote by S(t), where t is the curve parameter.

The vector field n is normal to S(t), so

$$n \cdot \xi_a = 0, \tag{4.1}$$

where ξ_a are the Killing vector fields which span the surfaces S(t) at each point p of S(t). On the other hand the vector field n is also invariant under the group of isometries, so

$$L_{\xi_a} n = 0. \tag{4.2}$$

I don't think so, for isn't it true that if n obeys ~~the~~ equations (4.1) and (4.2), so will $\lambda(t)\, n$ for any λ? This should be $n^i{}_{;j}\, n^j \propto n^i$. (I suggest you include the 2 line proof)

From equations (4.1) and (4.2) it can be shown that

$$\nabla_n n = n^i{}_{;j}\, n^j = 0, \tag{4.3}$$

which says that the integral curves of n are geodesics. The vector field n may then be defined by

$$n_i = t_{,i}, \quad n^i n_i = -1, \tag{4.4}$$

where t is the affine parameter on the geodesics with n as tangent vector. Thus the parameter t can be seen to be the proper time along the normals. The coordinates x^α are in general chosen constant along the normals:

$$x^\alpha{}_{,i} n^i = 0. \tag{4.5}$$

When the coordinates $(x^i) = (t, x^\alpha)$ are such that eqs. (4.4) and (4.5) are satisfied, we say that the system of coordinates form a <u>normalized</u> <u>comoving</u> <u>coordinate</u> <u>system</u>. I would call this a "synchronous" system, leaving the term "comoving" to be associated with the matter velocity fields.

From the Killing vector field basis (ξ_a) we can construct another basis (E_a) (see chapter 3, section 3.2) that spans the tangent space T_p at every point p of S(t) such that

$$L_{\xi_a} E_b = [\xi_a, E_b] = 0. \tag{4.6}$$

The vector fields (E_a) are such that, if we define the metric

$$g_{ab} = E_a \cdot E_b, \tag{4.7}$$

then we have by virtue of their invariance, eq.(4.6), that

$$L_\xi \, g_{ab} = 0. \tag{4.8}$$

Thus the scalar product defined in eq.(4.7) is constant on each surface of homogeneity S(t), t = const. Therefore it can only be an explicit function of the proper time t; $g_{ab} = g_{ab}(t)$.

In particular, when the basis (E_a) is chosen to be invariant along the normals we have

$$L_n \, E_a = [n, E_a] = 0; \tag{4.9}$$

then, by virtue of equation (4.4) and equation (4.9), we have that

$$\partial_\alpha E_a = 0. \tag{4.10}$$

So in this case the basis of vector fields (E_a) are time independent.

Thus, the line element for a spatially homogeneous space-time may be written as

$$ds^2 = - dt^2 + g_{\alpha\beta}(t)E^{\alpha}{}_{i}E^{\beta}{}_{j}dx^{i}dx^{j}, \tag{4.11}$$

where $\{E^{\alpha}{}_{i}dx^{i}\}$ is the 1-form basis, dual to $\{E_{\alpha}\}$, spaning the tangent space of S(t).

4.3 Bianchi Universes

The space-times which have a 3-dimensional group of isometries acting simply transitively on the surfaces S(t) are called <u>Bianchi Universes</u>, after L. Bianchi (1897) who was the first to give a classification for the 3-dimensional Lie Algebras.

The Killing vector fields $\{\xi_a, a = 1,2,3\}$ are the generators of the group of isometries. They form a Lie algebra (see chapter 3, section 3.2), that is

$$[\xi_{\alpha}, \xi_{\beta}] = C^{\gamma}{}_{\alpha\beta}\xi_{\gamma}, \tag{4.12}$$

where the C's are the <u>structure constants</u> of the Lie group of isometries.

The classification of the 3-dimensional Lie algebras was first done by Bianchi (1897) based on the derived algebras of a G_3 (see definiton 3.10). His classification gives 9 independent Lie algebra structures for a G_3; it is as follows:

Dimension of the Derived algebra -	Group type
0	I
1	II, III
2	IV, V, VI, VII
3	VIII, IX

A new classification for a G_3 has recently been given by Estabrook, Walquist and Behr (1968) and also by Ellis and MacCallum (1969) based on the invariants of the decomposition of the structure constants $C^{\alpha}{}_{\beta\gamma}$ (as given in Heckmann and Schucking, 1962).

The structure constants are anti-symmetric in the last pair of indices (eq. 3.8), thus we are left with 9-parameters which may be written in terms of a 3x3 matrix $N^{\alpha\beta}$ as

type as ½ or put in parentheses

$$(1/2)\,\epsilon^{\alpha\beta\gamma}C^{\mu}{}_{\alpha\beta} = N^{\mu\gamma}, \tag{4.13}$$

where $\epsilon^{\alpha\beta\gamma}$ are the components of the totally anti-symmetric pseudo-tensor with $\epsilon^{123} = (\det g_{\alpha\beta})^{-1/2}$. The matrix $N^{\alpha\beta}$ may be decomposed into its symmetric and anti-symmetric parts, $N^{(\alpha\beta)}$ and $N^{[\alpha\beta]}$ respectively. The anti-symmetric part in its turn may be written in terms of a covector a_λ, resulting in

$$N^{\alpha\beta} = n^{\alpha\beta} + \epsilon^{\alpha\beta\lambda}a_\lambda, \tag{4.14}$$

where $n^{\alpha\beta} = N^{(\alpha\beta)}$.

If we now substitute eq.(4.14) into eq.(4.13) we obtain an expression for the structure constants in terms of the symmetric matrix $n^{\alpha\beta}$ and the covector a_λ given by

$$C^{\alpha}{}_{\beta\gamma} = \epsilon_{\beta\gamma\lambda}n^{\alpha\lambda} + \delta^{\alpha}{}_{\gamma}a_{\beta} - \delta^{\alpha}{}_{\beta}a_{\gamma}. \tag{4.15}$$

The vector a_λ and the symmetric pseudo-tensor $n^{\alpha\beta}$ (of density -½) can be written directly from the C's as

$$a_\lambda = (1/2)C^\alpha{}_{\lambda\alpha}, \tag{4.16}$$

$$n^{\alpha\beta} = (1/2)\epsilon^{\mu\nu(\alpha}C^{\beta)}{}_{\mu\nu}. \tag{4.17}$$

The structure constants satisfy the Jacobi identities, eq.(3.9), which by use of the decomposition (4.15) reflects equivalently in terms of a_λ and $n^{\alpha\beta}$ as

$$n^{\alpha\beta}a_\beta = 0. \tag{4.18}$$

Actually, due to the anti-symmetry of the structure constants and the Jacobi identities, the C's consist only of 6 independent parameters.

The group classification of Ellis and MacCallum (1969) is made in terms of i) the rank and signature of $n^{\alpha\beta}$, ii) the vanishing or not of the vector a_λ and iii) a parameter h which is defined when $a_\lambda \neq 0$ by

seems inconsistent

$$a_\alpha a_\beta = (h/2)\,\epsilon_{\alpha\mu\nu}\epsilon_{\beta\lambda\sigma}n^{\mu\lambda}n^{\nu\sigma}, \tag{4.19}$$

which are the only invariant quantities under action of GL(3, R).

Those models in which $a_\lambda = 0$ are called <u>class A</u>, and those with $a_\lambda \neq 0$ are called <u>class B</u> models after Ellis and MacCallum (1969).

The pseudo-tensor $n^{\alpha\beta}$ can be diagonalized if we choose the Killing vector fields at a point p of S(t) to be the eigenvectors of $n^{\alpha\beta}$; the pseudo-tensor $n^{\alpha\beta}$ may then be written as

$$n^{\alpha\beta} = \mathrm{diag}(n_1, n_2, n_3). \tag{4.20}$$

However, due to the Jacobi identities, eqs.(4.18), one of the Killing vectors may be chosen parallel to the vector a_λ. If we take this Killing vector field to be ξ_3 (note that in Ellis and MacCallum, 1969, the choice was ξ_1), the vector a_λ is reduced to

$$a_\lambda = (0,\ 0,\ a) = a\delta^3{}_\lambda . \tag{4.21}$$

The form of $n^{\alpha\beta}$ and a_λ as given in equations (4.20) and (4.21) are said to be in the <u>standard diagonal form</u>. The Jacobi identities are now simply

$$an_3 = 0, \tag{4.22}$$

which shows that in class B models, since $a \neq 0$, we must have $n_3 = 0$. The rank of $n^{\alpha\beta}$ for class B models is thus always less or equal to 2. The parameter h in the standard diagonal form is given when $n_1 n_2 \neq 0$ by

$$h = a^2/n_1 n_2 . \tag{4.23}$$

In terms of the quantities n_i (i = 1, 2, 3) and a, the commutators of the Killing vector fields (eq. 4.12) are

$$\begin{aligned} [\xi_1,\ \xi_2] &= n_3\xi_3, \\ [\xi_2,\ \xi_3] &= n_1\xi_1 - a\xi_2, \\ [\xi_3,\ \xi_1] &= a\xi_1 + n_2\xi_2. \end{aligned} \tag{4.24}$$

The Killing vector fields can still be rescaled; this does not alter the invariant quantities: rank and signature of $n^{\alpha\beta}$, $a = 0$ or $a \neq 0$ or h. The rescaling can nevertheless force the components n_i and a to be ± 1 when non-zero for the case of $h = 0$ or $h = -1$, but when $h \neq 0$ or -1 the component a can only be reduced to $a = |h|^{1/2}$. In the following the group-type classification is given for the standard diagonal form with the <u>canonical values</u> of a and n_i.

Table 1

Group-type classification with canonical values of a_λ and $n^{\alpha\beta}$.

define in table caption

Class	Group type	rank of $n^{\alpha\beta}$	sign. of $n^{\alpha\beta}$	n_1	n_2	n_3	a	h	Dim. D
A	I	0	0	0	0	0	0	-	0
	II	1	+1	0	0	1	0	-	1
	VI_0	2	0	1	-1	0	0	0	2
	VII_0	2	+2	1	1	0	0	0	2
	VIII	3	+1	1	1	-1	0	-	3
	IX	3	+3	1	1	1	0	-	3
B	V	0	0	0	0	0	1	-	2
	IV	1	+1	1	0	0	1	-	2
	$III=VI_{h=-1}$	2	0	1	-1	0	1	-1	1
	$VI_{h(\neq 0,-1)}$	2	0	1	-1	0	a	$-a^2$	2
	$VII_{h(\neq 0)}$	2	2	1	1	0	a	a^2	2

From table 1 it is clear that in the cases of VI_h and VII_h there are an infinite family of Bianchi types because the parameter h assumes all values on the real line, $h \in R$.

positive?

4.4 Tetrad Basis

It is convenient at this stage to introduce a tetrad basis $\{ \mathbf{e}_a, a = 0,1,2,3\}$ such that

$$\mathbf{e}_0 \cdot \mathbf{e}_0 = -1, \tag{4.25}$$

and the 3 vectors $\{\mathbf{e}_\alpha\}$ are such that they span the tangent space at of $S(t)$, t = const., with

$$\mathbf{e}_0 \cdot \mathbf{e}_\alpha = 0. \tag{4.26}$$

That is the simplest choice that can be made: it corresponds to the case of a lapse function $N = 1$ and a shift vector $N^i = 0$, which is called the <u>synchronous</u> <u>gauge</u> (Jantzen, 1984). Condition (4.25) states the time-like character of the vector $\mathbf{e}_0$ and conditions (4.26) state the orthogonality of $\mathbf{e}_0$ with respect to the surface forming basis $\{\mathbf{e}_\alpha\}$.

An obvious choice for $\mathbf{e}_0$ is

$$\mathbf{e}_0 = n, \tag{4.27}$$

which will be adopted here. As the 3 vectors lie on the surface $S(t)$, t = const., we have

$$\mathbf{e}_\alpha = \mathbf{v}_\alpha{}^\beta(x^i)\boldsymbol{\xi}_\beta. \tag{4.28}$$

It can be shown that the basis $\{ \mathbf{e}_\alpha\}$ may be chosen at a point p of $S(t)$ (see section 3.2) such that

$$L_{\xi_\alpha} e_\beta = [\xi_\alpha, e_\beta] = 0, \tag{4.29}$$

that is, the vectors $\{\mathbf{e}_\alpha\}$ can be chosen invariant under the group of isometries.

Of course (4.29) can also be written

$$[\xi_\alpha, e_a] = 0, \tag{4.30}$$

because $L_{\xi_\alpha} n = 0$.

The commutators of the tetrad basis (e_a) define the structure functions $\gamma^a{}_{bc}(x^a)$,

$$[e_a, e_b] = \gamma^c{}_{ab} e_c, \tag{4.31}$$

where the γ's satisfy equations (2.22 and 2.23). From eqs. (4.28 and 4.31) we get the result

$$\gamma^o{}_{o\alpha} = 0 = \gamma^o{}_{\alpha\beta}. \tag{4.32}$$

The Jacobi identity applied to the vectors (ξ_α, e_a, e_b) gives

$$\xi_\alpha(\gamma^a{}_{bc}) = 0, \tag{4.33}$$

which implies

$$\gamma^a{}_{bc} = \gamma^a{}_{bc}(t), \tag{4.34}$$

-the structure functions may depend only on time (MacCallum, 1973).

The structure functions γ's and the structure constants C's are seen to be related; from equations (4.12), (4.28) and (4.30) we get

$$\psi_\alpha{}^\mu \psi_\beta{}^\nu C^\lambda{}_{\mu\nu} = - \gamma^\sigma{}_{\alpha\beta} \psi_\sigma{}^\lambda, \tag{4.35}$$

where $\psi^\alpha{}_\beta \, \psi_\alpha{}^\lambda = \delta_\beta{}^\lambda$. One particular case occurs when the basis $\{e_\alpha\}$ is also invariant along the integral curves of $n = e_o$, that is, when

$$[e_o, e_\alpha] = 0. \tag{4.36}$$

In this case we have the result of equation (4.10), so

$$\gamma^a{}_{ob} = 0. \tag{4.37}$$

The Jacobi identity applied to (e_o, e_α, e_β), is

$$\partial_o \gamma^\alpha{}_{\mu\nu} + \gamma^\alpha{}_{o\sigma}\gamma^\sigma{}_{\mu\nu} + \gamma^\alpha{}_{\nu\sigma}\gamma^\sigma{}_{o\mu} - \gamma^\alpha{}_{\mu\sigma}\gamma^\sigma{}_{o\nu} = 0. \tag{4.38a}$$

When eq. (4.37) holds, we conclude that

$$\partial_o \gamma^\alpha{}_{\mu\nu} = 0 \quad \Leftrightarrow \gamma^\alpha{}_{\mu\nu} = \text{const}. \tag{4.38b}$$

The vectors $\{e_\alpha\}$ form the Lie algebra of a group of transformations called the reciprocal group to the group of isometries (see section 3.3). If the functions $\Psi_\alpha{}^\beta$ at $p \in S(t)$ are chosen to be

$$\Psi_\alpha{}^\beta(p) = -\delta_\alpha{}^\beta, \qquad t = t_1 ? \tag{4.39}$$

equation (4.35) gives

$$\gamma^\alpha{}_{\beta\lambda} = C^\alpha{}_{\beta\lambda}, \tag{4.40}$$

which holds everywhere on the orbits.

In like manner to the decomposition of the structure constants in terms of the pseudo-tensor $n^{\alpha\beta}$ and vector a_λ, the structure functions γ's may be written

$$\gamma^\alpha{}_{\mu\nu} = \epsilon_{\mu\nu\lambda}N^{\alpha\lambda} + \delta^\alpha{}_\nu A_\mu - \delta^\alpha{}_\mu A_\nu, \tag{4.41}$$

with the difference that here $N^{\alpha\beta} = N^{\alpha\beta}(t)$ and $A_\beta = A_\beta(t)$. When equation (4.39) is valid on $S(t)$, $t = t_1$, then

$$N^{\alpha\beta}(t=t_1) = n^{\alpha\beta}(t=t_1), \quad A_\beta(t=t_1) = a_\beta(t=t_1), \tag{4.42}$$

although they will not, in general, be valid for all t. Equation (4.42) remains valid at all times t in the special case that eq. (4.36) is fulfilled everywhere. Therefore, the standard diagonal form and the canonical values of Table 1 can always be given to $N^{\alpha\beta}$ and A_β on the surface $S(t_1)$, being valid everywhere in the spacetime when $[e_0, e_\alpha] = 0$.

The space-time metric is then given by

$$ds^2 = -dt^2 + g_{\alpha\beta}(t)e^\alpha{}_i e^\beta{}_j dx^i dx^j, \tag{4.43}$$

where the $(e^\alpha{}_i)$ are functions of the spatial coordinates x^λ only.

For the Bianchi types of Table 1, the Killing vector fields (ξ_α), the corresponding group invariant bases (e_α) and their 1-form duals are listed in Jantzen (1979) pp. 217, and also in MacCallum (1979a, 1979b).

CHAPTER 5

DYNAMICS OF THE BIANCHI UNIVERSES

5.1 Invariant Basis Freedom

The invariant basis $\{e_\alpha\}$ has the freedom to be transformed under a time-dependent linear transformation $\Lambda_\alpha{}^\beta(t)$ which preserves equations (4.24), (4.25) and (4.29). The transformed tetrad basis is then

$$e_{0'} = e_0, \tag{5.1a}$$

$$e_{\alpha'} = \Lambda_{\alpha'}{}^\alpha e_\alpha, \tag{5.1b}$$

where $\Lambda_{\alpha'}{}^\alpha$ is non-singular, that is, $\det(\Lambda_{\alpha'}{}^\alpha) \neq 0$.

Under the transformations (5.1) the metric tensor $g_{ab} = e_a . e_b$ changes to

$$g_{0'0'} = e_{0'} . e_{0'} = e_0 . e_0 = g_{00}, \tag{5.2a}$$

$$g_{0'\alpha'} = e_{0'} . e_{\alpha'} = \Lambda_{\alpha'}{}^\alpha e_0 . e_\alpha = \Lambda_{\alpha'}{}^\alpha g_{0\alpha}, \tag{5.2b}$$

$$g_{\alpha'\beta'} = e_{\alpha'} . e_{\beta'} = \Lambda_{\alpha'}{}^\alpha \Lambda_{\beta'}{}^\beta e_\alpha . e_\beta = \Lambda_{\alpha'}{}^\alpha \Lambda_{\beta'}{}^\beta g_{\alpha\beta}, \tag{5.2c}$$

with $g_{0'0'} = -1$ and $g_{0'\alpha'} = 0$.

The structure functions transform under (5.1) as follows: Let

$$[e_{a'}, e_{b'}] = \gamma^{c'}{}_{a'b'} e_{c'}, \tag{5.3}$$

where the $\gamma^{c'}{}_{a'b'}$ are then the structure functions for the transformed basis $\{e_{a'}\}$. Then from equations (5.1), (5.3), (4.30) and (4.31) we have

$$\gamma^{o'}{}_{a'b'} = 0, \tag{5.4a}$$

$$\gamma^{\alpha'}{}_{o'\beta'} = \partial_o(\Lambda_{\beta'}{}^{\alpha})\Lambda^{\alpha'}{}_{\alpha} + \Lambda_{\beta'}{}^{\beta}\gamma^{\alpha}{}_{o\beta}\Lambda^{\alpha'}{}_{\alpha}, \tag{5.4b}$$

$$\gamma^{\alpha'}{}_{\beta'\lambda'} = \Lambda_{\beta'}{}^{\beta}\Lambda_{\lambda'}{}^{\lambda}\gamma^{\alpha}{}_{\beta\lambda}\Lambda^{\alpha'}{}_{\alpha}. \tag{5.4c}$$

If the invariant basis $\{e_a\}$ is dragged along the normals, equations (4.36) are valid. This leads to

$$\gamma^{\alpha'}{}_{o'\beta'} = \partial_o(\Lambda_{\beta'}{}^{\alpha})\Lambda^{\alpha'}{}_{\alpha}, \tag{5.5a}$$

$$\gamma^{\alpha'}{}_{\beta'\lambda'} = \Lambda_{\beta'}{}^{\beta}\Lambda_{\lambda'}{}^{\lambda}\gamma^{\alpha}{}_{\beta\lambda}\Lambda^{\alpha'}{}_{\alpha}, \tag{5.5b}$$

where $\gamma^{\alpha}{}_{\beta\lambda} = C^{\alpha}{}_{\beta\lambda}$ (see eq. 4.39). Therefore the transformed basis is also invariant along the normals if and only if

$$\partial_o(\Lambda_{\beta'}{}^{\alpha}) = 0 \iff \Lambda_{\beta}{}^{\alpha} = \text{const.} \tag{5.7}$$

It is obvious that the basis $\{e_{a'}\}$ is an invariant basis;

$$[\xi_\lambda, e_{o'}] = [\xi_\lambda, e_o] = 0, \tag{5.8a}$$

$$[\xi_\lambda, e_{\alpha'}] = [\xi_\lambda, \Lambda_{\alpha'}{}^{\alpha}(t)e_\alpha] = \Lambda_{\alpha'}{}^{\alpha}(t)[\xi_\lambda, e_\alpha] = 0. \tag{5.8b}$$

From the definition (4.40), the pseudo-tensor $N^{\alpha\beta}$ and the vector A_β transform under $\Lambda_{\alpha'}{}^{\alpha}(t)$ as (Jantzen,1979)

$$N^{\alpha'\beta'} = \det(\Lambda_{\lambda'}{}^{\lambda})\Lambda^{\alpha'}{}_{\alpha}N^{\alpha\beta}\Lambda^{\beta'}{}_{\beta}, \tag{5.9a}$$

$$A_{\beta'} = \Lambda_{\beta'}{}^{\beta}A_\beta. \tag{5.9b}$$

The determinant of $\Lambda_{\lambda'}{}^{\lambda}$ appears in eq. (5.9a) due to the pseudo-tensor character of $N^{\alpha\beta}$. It is clear from (5.9a) that $N^{\alpha\beta}$ transforms like a tensor when the transformations $\Lambda_{\lambda'}{}^{\lambda}$ are restricted to the subgroup of transformations of the Lorentz group with determinant +1, that is, the special Lorentz transformations.

When the basis $\{e_a\}$ is invariant along the normals, i.e., when $[e_0, e_\alpha] = 0$, we have (see eqs. 4.41)

$$N^{\alpha\beta} = n^{\alpha\beta}, \tag{5.10a}$$

$$A_\beta = a_\beta, \tag{5.10b}$$

being valid everywhere in the space-time. However, the transformed quantities associated with them via eqs. (5.9) will be in general time-dependent.

5.2 Dynamical Variables

The Einstein field equations determine the dynamics of the space-time. As we have already seen, in Bianchi Universes the metric $g_{ab}(t)$ and the structure functions $\gamma^a{}_{bc}(t)$ are the essential quantities of the system; they are the <u>dynamical variables</u>. Of course we have to keep in mind the fluid variables as well; those will be called the <u>matter variables</u> (see chapter 2, section 2.4).

An enormous simplification of the Einstein field equations for the Bianchi Universes is achieved when the metric and the structure functions are simultaneously put in the simplest possible form. The <u>first approach</u> towards simplification, which has been adopted by many authors (cf. Ellis, 1967), is to choose the invariant basis $\{e_a\}$ to be an orthonormal tetrad basis everywhere in the space-time, that is,

$$g_{\alpha\beta} = e_\alpha . e_\beta = \delta_{\alpha\beta}. \tag{5.11}$$

Thus in the first approach the dynamical variables are reduced to the structure functions $\gamma^a{}_{bc}(t)$. In this case no transformation can reduce the γ's at all times to the canonical form of table 1; in fact the transformations preserving the orthonormality of the basis $\{e_a\}$ can, at most, involve the commutator (MacCallum, 1979a, 1979b)

$$[e_0, e_\alpha] = -\theta_\alpha e_\alpha, \text{ (no sum on } \alpha). \tag{5.12}$$

The <u>second approach</u> towards simplification of the Einstein field equations is concerned with reducing the structure functions $\gamma^a{}_{bc}(t)$ to the simplest possible form. This can be achieved by choosing the invariant basis $\{e_a\}$ to satisfy

$$[e_0, e_\alpha] = 0, \tag{5.13}$$

which implies that $\gamma^\alpha{}_{\beta\lambda} = C^\alpha{}_{\beta\lambda}$. In this case the dynamical variables are the metric components $g_{ab}(t)$ (see Taub, 1951).

The explicit form of the Einstein field equations for the first approach can be found in Ellis and MacCallum (1969), and for the second approach in MacCallum (1973, 1979b).

In the previous two approaches, either the metric or the structure functions were simplified <u>individually</u> as much as possible. Nevertheless further simplification of the field equations of the Bianchi Universes may still be attained if we preserve the virtue of the second approach and in addition simplify the form of the metric. This can be done if we make use of the time-dependent transformations $\Lambda_\alpha{}^\beta(t)$ described in section 5.1.

The <u>third approach</u> arises when the time-dependent transformations $\Lambda_\alpha{}^\beta(t)$ are such that they preserve the structure constants of the Bianchi Lie algebras. This allows some of the metric time-dependence to be absorbed by some of the parameters of the transformations leading to a simpler metric form. Although this procedure simplifies the metric, the price that must be paid for it is that the new frame is no longer invariant along the normals due to the first term on the right of eq. (5.4b), which means that we now have the $\gamma^{\alpha'}{}_{0'\beta'} \neq 0$ for some α' and β'. The Jacobi identities (4.37a) give

$$\gamma^{\alpha'}{}_{0'\sigma'}C^{\sigma'}{}_{\mu'\nu'} + \gamma^{\sigma'}{}_{0'\mu'}C^{\alpha'}{}_{\nu'\sigma'} - \gamma^{\sigma'}{}_{0'\nu'}C^{\alpha'}{}_{\mu'\sigma'} = 0, \tag{5.14}$$

which are the integrability conditions relating the $\gamma^{\alpha'}{}_{0'\beta'}$ to the structure constants $C^{\alpha'}{}_{\beta'\lambda'}$.

It is this third approach that will be investigated in detail in this thesis. In the past some authors (Collins and Hawking, 1973; Bogoyalenski and Novikov, 1975) have used this approach to investigate particular Bianchi types. Recently, Siklos (1980), in a short letter, attempted to extend this approach to class B Bianchi Universes. However his results are in some points incorrect and incomplete. Jantzen (1979, 1983) has also used the time-dependent transformations Λ's in his extensive Hamiltonian Cosmology program.

5.3 The Automorphism Group

The third approach relies on the set of transformations $\Lambda_\alpha{}^\beta(t)$ that leave the structure constants of the Bianchi Lie algebras invariant. These transformations form a subgroup of GL(3,R) which is called the <u>automorphism group</u> of the Lie algebra. Formally, the automorphism group, denoted here by AUT(T) where T stands for the Bianchi type in consideration, is the set of transformations that leaves the transformed structure functions $\gamma^{\alpha'}{}_{\beta'\lambda'}$ numerically identical to $\gamma^\alpha{}_{\beta\lambda}$, in other words,

$$\gamma^{\alpha'}{}_{\beta'\lambda'} = \delta^\beta{}_{\beta'}\delta^\lambda{}_{\lambda'}\gamma^\alpha{}_{\beta\lambda}\delta^{\alpha'}{}_\alpha . \tag{5.15}$$

OK

do you mean a Kronecker δ here?

As the structure functions can be split into the vector A_β and the pseudo-tensor $N^{\alpha\beta}$ ~~like~~ as in eq. (4.40), the automorphism group leaves these quantities invariant, that is,

$$A_{\beta'} = \delta_{\beta'}{}^{\beta} a_{\beta}, \tag{5.16a}$$

$$N^{\alpha'\beta'} = \delta^{\alpha'}{}_{\alpha} N^{\alpha\beta} \delta^{\beta'}{}_{\beta}. \tag{5.16b}$$

The set of eqs. (5.16) are obviously equivalent to eqs. (5.15). The <u>special automorphism group</u> is the subgroup of the automorphism group specified by the condition that its elements have determinant +1. Under the action of the special automorphism, pseudo-tensors transform as tensors (for example, $N^{\alpha\beta}$ transform as a second order tensor in eq. (5.9a) because $\det(A_{\lambda'}{}^{\lambda}) = 1$).

Let C be the submanifold of $R^3 X R^3 X R^3$ consisting of all structure constants $C^\alpha{}_{\beta\lambda}$ satisfying eq. (4.2). Then the automorphism group may be interpreted as the stability group of the action of GL(3,R) on C, while the different group types correspond to the different orbits O under the action of GL(3,R) on C. In other words the automorphism group leaves the group structure constants fixed while points in the same orbits represent the same group type although with different structure constants.

The automorphism group for each Bianchi type can be found easily for a basis given in the canonical form of table 1. Harvey (1979) has given the automorphism group for each Bianchi type based on the Taub (1951) group classification, where he has made direct use of the eqs. (5.5b). However the calculation is much simpler if done in terms of the classification presented in table 1 by use of the eqs. (5.9). In what follows the automorphism group, AUT(T), and the special automorphism group, SAUT(T), are given for each Bianchi type in accordance with Table 1.

Class A models: $a_\beta = 0$

Type I: $n^{\alpha\beta} = 0$

$AUT(I) = GL(3,R)$

$SAUT(I) = SL(3,R)$

Type II: $n^{\alpha\beta} = (0, 0, 1)$

$AUT(II) = \{ A \in GL(3,R);\ A_3{}^1 = A_3{}^2 = 0,\ A_3{}^3 = A_1{}^1 A_2{}^2 - A_1{}^2 A_2{}^1 \}$

$SAUT(II) = \{ A \in AUT(II);\ A_3{}^3 = \pm 1 \}$

Type VI_0: $n^{\alpha\beta} = (1, -1, 0)$

$AUT(VI_0) = \{ A \in GL(3,R);\ A_1{}^3 = A_2{}^3 = 0,\ A_1{}^1 = A_2{}^2,\ A_2{}^1 = A_1{}^2,\ A_3{}^3 = 1 \}$

$SAUT(VI_0) = \{ A \in AUT(VI_0);\ (A_1{}^1)^2 - (A_1{}^2)^2 = 1 \}$

Type VII_0: $n^{\alpha\beta} = (1, 1, 0)$

$AUT(VII_0) = \{ A \in GL(3,R);\ A_1{}^3 = A_2{}^3 = 0,\ A_1{}^1 = A_2{}^2,\ A_1{}^2 = -A_2{}^1,\ A_3{}^3 = 1 \}$

$SAUT(VII_0) = \{ A \in AUT(VI_0);\ (A_1{}^1)^2 + (A_1{}^2)^2 = 1 \}$

Type VIII: $n^{\alpha\beta} = (1, 1, -1)$

$AUT(VIII) = SO(2,1)$, this is the 3-dimensional special Lorentz group generated by two boosts and one rotation.

$SAUT(VIII) = SO(2,1)$

Type IX: $n^{\alpha\beta} = (1, 1, 1)$

$AUT(IX) = SO(3,R)$, that is the 3-dimensional special Lorentz group generated by three rotations.

$SAUT(IX) = SO(3,R)$

Class B models: $a_\beta \neq 0$

<u>Type V</u>: $n^{\alpha\beta} = 0$

$AUT(V) = \{ A \in GL(3,R);\ A_1{}^3 = A_2{}^3 = 0,\ A_3{}^3 = 1\}$

$SAUT(V) = \{ A \in AUT(V);\ A_1{}^1A_2{}^2 - A_1{}^2A_2{}^1 = 1\}$

<u>Type IV</u>: $n^{\alpha\beta} = (1,\ 0,\ 0),\ a = 1$

$AUT(IV) = \{ A \in GL(3,R);\ A_1{}^2 = A_1{}^3 = A_2{}^3 = 0,\ A_1{}^1 = A_2{}^2,\ A_3{}^3 = 1\}$

$SAUT(IV) = \{ A \in AUT(IV);\ A_1{}^1 = \pm 1\}$

<u>Type VI_h</u>: $n^{\alpha\beta} = (1,\ -1,\ 0),\ a^2 = -h$

$AUT(VI_h) = AUT(VI_o)$

$SAUT(VI_h) = SAUT(VI_o)$

<u>Type VII_h</u>: $n^{\alpha\beta} = (1,\ 1,\ 0),\ a^2 = h$

$AUT(VII_h) = AUT(VII_o)$

$SAUT(VII_h) = SAUT(VII_o)$

The parameter h defined in eq. (4.19) is invariant under the action of GL(3,R); this explains why the automorphism group of Bianchi types VI_h and VII_h are identical to $AUT(VI_o)$ and $AUT(VII_o)$, respectively. For the same reason the automorphism group of class A type $VI_{h=-1/9}$ and class B type $III = VI_{h=-1}$ are identical to $AUT(VI_o)$.

5.4 Automorphism Families

It is convenient to write the automorphism group for each Bianchi type in terms of a family of 1- or 2-parameter~~s~~ abelian transformations which individually leave the group type structure invariant. In what follows we give the set of such transformations and point out which canonical structure constants of table 1 are invariant under each of them. The full 9-dimensional general linear group, GL(3,R), may be generated by this set of parametrized transformations.

1-Parameter Unimodular Transformation A_1

$$A_{1\alpha}{}^{\beta} = \begin{pmatrix} \cos\alpha(t) & \sin\alpha(t) & 0 \\ -\sin\alpha(t) & \cos\alpha(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.17}$$

These are rotations about the 3-axis. From eqs. (4.23), (5.1b) and (5.17) we immediately see that the canonical structure constants are invariant iff $n_1 = n_2$. Therefore, the 1-parameter transformations (5.17) leave invariant the Bianchi types I, II, VII_0, VIII, IX and VII_h (see table 1). The metric components change by this transformation to:

$$g_{1'1'} = \cos^2\alpha\ g_{11} + 2\sin\alpha\cos\alpha\ g_{12} + \sin^2\alpha\ g_{22}$$
$$g_{1'2'} = -\sin\alpha\cos\alpha\ g_{11} + (\cos^2\alpha - \sin^2\alpha)g_{12} + \sin\alpha\cos\alpha\ g_{22}$$
$$g_{1'3'} = \cos\alpha\ g_{13} + \sin\alpha\ g_{23}$$
$$g_{2'2'} = \sin^2\alpha\ g_{11} - \sin\alpha\cos\alpha\ g_{12} + \cos^2\alpha\ g_{22}$$
$$g_{2'3'} = -\sin\alpha\ g_{13} + \cos\alpha\ g_{23}$$

$g_{3'3'} = g_{33}$

1-Parameter Unimodular Transformations A_2

$$A_{2\alpha}{}^{\beta} = \begin{pmatrix} \cosh \phi(t) & -\sinh \phi(t) & 0 \\ -\sinh \phi(t) & \cosh \phi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.18}$$

These are boosts in the (1-2) plane. Via the same procedure used for (5.17), we find that the canonical structure constants are invariant under (5.18) iff $n_1 + n_2 = 0$. This condition leads to the following Bianchi types being invariant: I, II, VI_o, V, III and VI_h. The metric components change by this transformation to:

$g_{1'1'} = \cosh^2\phi \; g_{11} - 2\sinh\phi \cosh\phi \; g_{12} + \sinh^2\phi \; g_{22}$

$g_{1'2'} = -\sinh\phi \cosh\phi \; g_{11} + (\sinh^2\phi + \cosh^2\phi) g_{12} - \sinh\phi \cosh\phi \; g_{22}$

$g_{1'3'} = \cosh\phi \; g_{13} - \sinh\phi \; g_{23}$

$g_{2'2'} = \sinh^2\phi \; g_{11} - 2\sinh\phi \cosh\phi \; g_{12} + \cosh^2\phi \; g_{22}$

$g_{2'3'} = -\sinh\phi \; g_{13} + \cosh\phi \; g_{23}$

$g_{3'3'} = g_{33}$

It is also useful to consider a transformation $A_{2'}$ instead of A_2.

1-Parameter Unimodular Transformation $A_{2'}$

$$A_{2'\alpha}{}^{\beta} = \begin{pmatrix} e^{-\phi(t)} & & \\ & e^{\phi(t)} & \\ & & 1 \end{pmatrix} \tag{5.19}$$

The canonical structure constants are invariant under $A_{2'}$ iff $n_1 = n_2 = 0$.

This leaves the following Bianchi types invariant: I, II and V. The metric components changed by this transformation are:

$$g_{1'1'} = e^{-\phi} g_{11}$$
$$g_{2'2'} = e^{2\phi} g_{22}$$
$$g_{1'3'} = e^{-\phi} g_{13}$$
$$g_{2'3'} = e^{\phi} g_{23}$$

1-Parameter Unimodular Transformation A_3

$$A_{3\alpha}{}^{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ -b_3(t) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.20}$$

These are null rotations about the 1-direction in the (1-2) plane. The canonical structure constants are invariant under A_3 iff $n_3 = 0$. This leaves the following Bianchi types invariant: I, II, V and IV. The metric components changed by this transformation are:

$$g_{1'2'} = g_{12} - b_3 g_{11}$$
$$g_{2'2'} = g_{22} - 2b_3 g_{12} + b_3{}^2 g_{11}$$
$$g_{2'3'} = g_{23} - b_3 g_{12}$$

2-Parameters Unimodular Transformations A_4

$$A_{4\alpha}{}^{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b_1(t) & -b_2(t) & 1 \end{pmatrix} \tag{5.21}$$

These are null rotations about the 3-direction. The canonical structure constants are invariant under A_4 iff $n_3 = 0$. This shows that Bianchi types I, VI_0, VII_0 and all of class B types are invariant under it. The metric components changed by this transformation are:

$$g_{1'3'} = g_{13} - b_1 g_{11} - b_2 g_{12}$$

$$g_{2'3'} = g_{23} - b_1 g_{12} - b_2 g_{22}$$

$$g_{3'3'} = g_{33} - 2b_1 g_{13} - 2b_2 g_{23} + 2b_1 b_2 g_{12} + b_1^2 g_{11} + b_2^2 g_{22}$$

1-Parameter Unimodular Transformations A_5

$$A_{5\alpha}{}^{\beta} = \begin{pmatrix} 1 & 0 & -b_5(t) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.22}$$

These are null rotations about the 3-direction in the (1-3)-plane. The canonical structure constants are invariant under A_5 iff $n_1 = a = 0$. This leaves the Bianchi types I and II invariant. The metric components changed by this transformation are:

$$g_{1'1'} = g_{11} - 2b_5 g_{13} + b_5^2 g_{33}$$

$$g_{1'2'} = g_{12} - b_5 g_{23}$$

$$g_{1'3'} = g_{13} - b_5 g_{33}$$

1-Parameter Unimodular Transformations A_6

$$A_{6\alpha}{}^{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -b_6(t) \\ 0 & 0 & 1 \end{pmatrix} \tag{5.23}$$

These are null rotations about the 3-direction in the (2-3)-plane. The canonical structure constants are invariant under A_6 iff $n_2 = a = 0$. The Bianchi types invariant under A_6 are thus types I and II. The metric components changed by this transformation are:

$$g_{1'2'} = g_{12} - b_6 g_{13}$$

$$g_{2'2'} = g_{22} - 2b_6 g_{23} + b_6^{\,2} g_{33}$$

$$g_{2'3'} = g_{23} - b_6 g_{33}$$

1-Parameter Non-Unimodular Transformations A_7

$$A_{7\alpha}{}^{\beta} = \begin{pmatrix} e^{-\lambda(t)} & & \\ & e^{-\lambda(t)} & \\ & & 1 \end{pmatrix} \tag{5.24}$$

These are conformal transformations in the (1-2)-plane. The canonical structure constants are invariant under A_7 iff $n_3 = 0$. The Bianchi types invariant under A_7 are: I, VI_o, VII_o and all of class B types. The metric components changed by this transformation are:

$$g_{1'1'} = e^{-2\lambda} g_{11}$$

$$g_{1'2'} = e^{-2\lambda} g_{12}$$

$$g_{1'3'} = e^{-\lambda} g_{13}$$

$$g_{2'2'} = e^{-2\lambda} g_{22}$$

$$g_{2'3'} = e^{-\lambda} g_{23}$$

1-Parameter Non-Unimodular Transformations A_8

$$A_{8\alpha}{}^{\beta} = \begin{pmatrix} e^{\rho(t)} & & \\ & e^{\rho(t)} & \\ & & e^{2\rho(t)} \end{pmatrix} \tag{5.25}$$

These are conformal transformations in all the three directions. The canonical structure constants are invariant under A_8 iff $a = 0$ and $n_1 = n_2 = 0$. This leaves only Bianchi types I and II invariant under A_8. The metric components changed by this transformation are:

$$g_{1'1'} = e^{-2\rho} g_{11}$$
$$g_{1'2'} = e^{-2\rho} g_{12}$$
$$g_{1'3'} = e^{-3\rho} g_{13}$$
$$g_{2'2'} = e^{-2\rho} g_{22}$$
$$g_{2'3'} = e^{-3\rho} g_{23}$$
$$g_{3'3'} = e^{-4\rho} g_{33}$$

The special automorphism groups are generated by the A_i ($i = 1$ to 6) as these are the unimodular transformations. The invariant family for each Bianchi type is given in the table 2. There is also given the automorphism group dimension s, which essentially corresponds to the number of different bases that we can choose for the Lie algebra of a group type with the same canonical form for the structure constants; and the orbit dimension d, which corresponds to the degree of freedom in expressing the structure constants of a particular group type. As we have seen in section 3.3, these quantities are related by $s + d = 9$, as $GL(3,R)$ is a 9-parameter group.

Table 2

Automorphism Invariant Family, Automorphism Group Dimension (s), and

Group Type Orbit Dimension (d) for each Bianchi type

Class	Type	Inv. family	s	d
A	I	GL(3,R)	9	0
	II	$A_1, A_2, A_3, A_5, A_6, A_8$	6	3
	VI_o	A_2, A_4, A_7	4	5
	VII_o	A_1, A_4, A_7	4	5
	VIII	SO(2,1)	3	6
	IX	SO(3,R)	3	6
B	V	A_1, A_2, A_3, A_4, A_7	6	4
	IV	A_3, A_4, A_7	4	5
	III	A_2, A_4, A_7	4	5
	VI_h	A_2, A_4, A_7	4	5
	VII_h	A_1, A_4, A_7	4	5

The family of transformations of a Bianchi type may be parametrized in a different way from that presented in Table 2. As an example we may choose for type V the transformation A_2' instead of A_2. The only restriction on such transformations is that they must be an automorphism generator and, obviously, linearly independent of the other transformations in a family.

CHAPTER 6

THE FIELD EQUATIONS FOR BIANCHI UNIVERSES

The time-dependent automorphism group can be used as a tool to simplify the metric of a Bianchi Universe while preserving its group structure. Some of the time-dependence of the metric may be absorbed by the automorphism parameters; as a consequence a simplification in their field equations is obtained. In this chapter we show how this mechanism works explicitly using the Bianchi type VII_h Universe for our computation.

6.1 The Field Equations

In the previous chapter we have shown how the metric $g_{ab}(t)$ and the structure functions $\gamma^a{}_{bc}(t)$ change under a non-singular time-dependent transformation $\Lambda_\alpha{}^\beta(t)$. In the third approach the metric components are (we have dropped the primes for sake of simplicity)

$$g_{oo} = -1, \; g_{o\alpha} = 0, \; g_{\alpha\beta} = g_{\alpha\beta}(t), \tag{6.1}$$

and the structure functions are

$$\gamma^o{}_{o\alpha} = 0, \quad \gamma^o{}_{\alpha\beta} = 0, \tag{6.2a}$$

$$\gamma^\alpha{}_{o\beta} = \gamma^\alpha{}_{o\beta}(t), \quad \gamma^\alpha{}_{\beta\lambda} = C^\alpha{}_{\beta\lambda} \text{ (constants).} \tag{6.2b}$$

The inverse metric components are

$$g^{oo} = -1, \quad g^{o\alpha} = 0, \quad g^{\alpha\beta} = g^{\alpha\beta}(t), \tag{6.3}$$

where $g_{\alpha\beta}g^{\beta\lambda} = \delta_\alpha{}^\lambda$ (see eq. 2.4).

The Einstein field equations with vanishing cosmological constant are given by (see section 2.4)

$$R_{ab} = T_{ab} - (1/2)Tg_{ab}. \tag{6.4}$$

To have eqs. (6.4) written explicitly in terms of the dynamical variables we should calculate the left and right side of these equations in terms of the dynamical variables. The Ricci tensor components are given by eq. (2.37) where the connections are given by eqs. (2.27). In the case under consideration the connections components are

$$\Gamma^o{}_{oo} = 0, \quad \Gamma^\alpha{}_{oo} = 0, \quad \Gamma^o{}_{o\alpha} = 0 = \Gamma^o{}_{\alpha o}, \tag{6.5a}$$

$$\Gamma^o{}_{\beta\delta} = K_{\beta\delta} - \gamma^\lambda{}_{o(\beta}g_{\delta)\lambda}, \tag{6.5b}$$

$$\Gamma^\epsilon{}_{o\beta} = K^\epsilon{}_\beta - (1/2)(\gamma^\epsilon{}_{o\beta} - g_{\beta\lambda}\gamma^\lambda{}_{o\alpha}g^{\alpha\epsilon}), \tag{6.5c}$$

$$\Gamma^\epsilon{}_{\beta o} = K^\epsilon{}_\beta - (1/2)(\gamma^\epsilon{}_{o\beta} + g_{\beta\lambda}\gamma^\lambda{}_{o\alpha}g^{\alpha\epsilon}) = 2g^{\epsilon\lambda}\Gamma^o{}_{\beta\lambda}, \tag{6.5d}$$

$$\Gamma^\epsilon{}_{\beta\lambda} = (1/2)(C^\epsilon{}_{\beta\lambda} + g^{\epsilon\alpha}C^\sigma{}_{\alpha(\beta}g_{\lambda)\sigma}), \tag{6.5e}$$

where $K_{\alpha\beta}$ is the <u>extrinsic</u> <u>curvature</u> defined by

$$K_{\alpha\beta} = (1/2)(L_n\, g)_{\alpha\beta} = (1/2)\partial_o g_{\alpha\beta} = (1/2)\dot{g}_{\alpha\beta}(t). \tag{6.6}$$

The Ricci tensor components are given by

$$R_{oo} = -\partial_o\Gamma^\alpha{}_{\alpha o} - \Gamma^\alpha{}_{\beta o}\Gamma^\beta{}_{\alpha o}$$

$$= -\dot{K} - K_{\alpha\beta}K^{\alpha\beta} + 2K^\alpha{}_\beta\gamma^\beta{}_{0\alpha} + \dot{\gamma}^\alpha{}_{o\alpha} - (1/2)\gamma^\alpha{}_{o\beta}\gamma^\beta{}_{o\alpha} -$$

$$-(1/2)g^{\delta\lambda}g_{\alpha\beta}\gamma^\alpha{}_{o\delta}\gamma^\beta{}_{o\lambda}, \tag{6.7a}$$

$$R_{0\alpha} = \Gamma^{\beta}{}_{\beta\lambda}\Gamma^{\lambda}{}_{\alpha 0} - \Gamma^{\lambda}{}_{\beta 0}\Gamma^{\beta}{}_{\lambda\alpha}$$

$$= C^{\beta}{}_{\beta\lambda}K^{\lambda}{}_{\alpha} + K^{\lambda}{}_{\beta}C^{\beta}{}_{\alpha\lambda} + (1/2)C^{\beta}{}_{\delta\beta}(\gamma^{\delta}{}_{0\alpha} + g^{\delta\lambda}g_{\alpha\sigma}\gamma^{\sigma}{}_{0\lambda}) -$$

$$- (1/2)(C^{\delta}{}_{\alpha\lambda}\gamma^{\lambda}{}_{0\delta} + g_{\epsilon\mu}g^{\beta\lambda}C^{\mu}{}_{\alpha\lambda}\gamma^{\epsilon}{}_{0\beta}), \tag{6.7b}$$

$$R_{\alpha\beta} = \partial_0\Gamma^{0}{}_{\beta\alpha} + \Gamma^{\lambda}{}_{\lambda 0}\Gamma^{0}{}_{\beta\alpha} - \Gamma^{0}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{0\beta} - \Gamma^{\lambda}{}_{0\alpha}\Gamma^{0}{}_{\lambda\beta} + \Gamma^{\lambda}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\beta\alpha} - \Gamma^{\lambda}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\lambda\beta}$$

$$= \dot{K}_{\alpha\beta} + (K - \gamma^{\lambda}{}_{0\lambda})(K_{\alpha\beta} - \gamma^{\lambda}{}_{0(\alpha}g_{\beta)\lambda}) - 2K_{\alpha\sigma}K^{\sigma}{}_{\beta} - 2\gamma^{\sigma}{}_{0(\alpha}K_{\beta)\sigma}$$

$$+ 2K^{\lambda}{}_{(\alpha}g_{\beta)\sigma}\gamma^{\sigma}{}_{0\lambda} - \dot{\gamma}^{\sigma}{}_{0(\alpha}g_{\beta)\sigma} + (1/2)\, g_{\sigma\lambda}\gamma^{\sigma}{}_{0\alpha}\gamma^{\lambda}{}_{0\beta} -$$

$$-(1/2)\, g_{\alpha\sigma}g_{\beta\mu}\gamma^{\sigma}{}_{0\lambda}\gamma^{\mu}{}_{0\delta}g^{\lambda\delta} + R^{*}{}_{\alpha\beta}, \tag{6.7c}$$

where

$$R^{*}{}_{\alpha\beta} = \Gamma^{\lambda}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\beta\alpha} - \Gamma^{\lambda}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\lambda\beta}$$

$$= C^{\lambda}{}_{\lambda\sigma}g^{\sigma\nu}C^{\mu}{}_{\nu(\alpha}g_{\beta)\mu} - C^{\lambda}{}_{\sigma\alpha}(C^{\sigma}{}_{\lambda\beta} + g_{\lambda\mu}g^{\sigma\delta}C^{\mu}{}_{\delta\beta}) +$$

$$+ (1/4)\, g_{\alpha\delta}g_{\beta\epsilon}C^{\delta}{}_{\sigma\nu}C^{\epsilon}{}_{\mu\lambda}g^{\sigma\mu}g^{\nu\lambda}. \tag{6.7d}$$

The dot `˙' means time derivative, $\dot{f} = \partial_t f$.

It follows that we can obtain an equation with no derivative of $K_{\alpha\beta}$:

$$R_{00} + g^{\alpha\beta}R_{\alpha\beta} = K(K - 2\gamma^{\lambda}{}_{0\lambda}) - K_{\alpha\beta}K^{\alpha\beta} + 2\gamma^{\beta}{}_{0\alpha}(K^{\alpha}{}_{\beta} - (1/4)\gamma^{\alpha}{}_{0\beta}) + (\gamma^{\lambda}{}_{0\lambda})^2 -$$

$$- 1/2\, g^{\epsilon\delta}g_{\beta\lambda}\gamma^{\beta}{}_{0\epsilon}\gamma^{\lambda}{}_{0\delta} + R^{*}, \tag{6.7e}$$

where

$$R^{*} = - g^{\alpha\beta}C^{\lambda}{}_{\lambda\alpha}C^{\sigma}{}_{\sigma\beta} - (1/2)g^{\alpha\beta}C^{\lambda}{}_{\sigma\alpha}(C^{\sigma}{}_{\lambda\beta} + 1/2\, g_{\lambda\mu}g^{\sigma\delta}C^{\mu}{}_{\delta\beta}). \tag{6.7f}$$

Note that for the second approach ($\gamma^{\alpha}{}_{0\beta} = 0$), the Ricci tensor components

(6.7) reduce to the form already presented in MacCallum (1979b) and in Wald (1984). (In MacCallum (1979b) the curvature tensor is defined as minus our definition here). The equation (6.7e) is independent of second derivatives of $g_{\alpha\beta}$ and first derivatives of $\gamma^{\alpha}{}_{o\beta}$.

To complete the field equations, the right hand side of (6.4) is required. Here the matter content is that of a perfect fluid with stress-energy ~~energy-momentum~~ tensor T_{ab} given by eq. (2.48). The matter variables are spatially-homogeneous quantities, that means they depend only on the time t. The 4-velocity of the fluid flow lines, u, is chosen to be tilted to the hypersurfaces of homogeneity S(t); it is related to the normals n by (King and Ellis, 1973)

$$u^a = \cosh\beta\ n^a + \sinh\beta\ c^a, \tag{6.8a}$$

where $\beta = \beta(t)$ is the tilt angle and $c^a = c^a(t)$ is a normal vector defined such that

$$c^a c_a = 1, \quad c_a n^a = 0. \tag{6.8b}$$

(We have dropped here the tilde on the c^a from that of King and Ellis, 1973).

For a perfect fluid eq. (6.4) is written as

$$R_{ab} = (\mu + p)u_a u_b + (1/2)(\mu - p)g_{ab}. \tag{6.9}$$

~~so,~~ From eqs. (6.8) and (6.9) we have

$$R_{oo} = (\mu + p)\cosh^2\beta + (1/2)(p - \mu), \tag{6.10a}$$

$$R_{o\alpha} = (\mu + p)\sinh\beta\ \cosh\beta\ c_\alpha, \tag{6.10b}$$

$$R_{\alpha\beta} = (1/2)(\mu - p)g_{\alpha\beta} + (\mu + p)\ \sinh^2\beta\ c_\alpha c_\beta. \tag{6.10c}$$

They imply

$$R_{oo} + g_{\alpha\beta}R^{\alpha\beta} = 2\mu + 2(\mu + p)\sinh^2\beta. \tag{6.10d}$$

The components of the normal vector n are given by

$$n_a = - \delta^o{}_a, \quad n^a = \delta^a{}_o, \; n_{a;b}n^b = 0. \tag{6.11}$$

Then, from eq. (6.8b) we immediately see that

$$c^o = 0 = c_o. \tag{6.12}$$

The conservations equations take the form (King and Ellis, 1973)

$$\partial_o[\ln(w\cosh\beta)] + n^a{}_{;a} + \tanh\beta \; c^a{}_{;a} = 0, \tag{6.13a}$$

$$\partial_o[\ln(r\sinh\beta)](\cosh\beta \; u^a + \sinh\beta \; c^a) + c^a{}_{;b}u^b + \cosh\beta \; n^a{}_{;b}c^b = 0, \tag{6.13b}$$

where we have used the definitions

$$w(t) = \exp \int (d\mu/dt)/(\mu + p) \; dt, \tag{6.14a}$$

$$r(t) = \exp \int (dp/dt)/(\mu + p) \; dt. \tag{6.14b}$$

If we assume an equation of state, $p = p(\mu)$, of the form $p = (\gamma - 1)\mu$, (see eq. 2.50), the quantities w(t) and r(t) are given by

$$w(t) = \mu(t)^{1/\gamma}, \tag{6.15a}$$

$$r(t) = \mu^{(\gamma - 1)/\gamma}. \tag{6.15b}$$

By virtue of eqs. (6.15) the conservation equations (6.13) become

$$\partial_o[\ln(w\cosh\beta)] + K - \gamma^\alpha{}_{o\alpha} + \tanh\beta \; C^\lambda{}_{\alpha\lambda}c^\lambda = 0, \tag{6.16a}$$

$$\partial_o[\ln(r\sinh\beta)]c^\alpha + \partial_o c^\alpha + 2K^\alpha{}_\lambda c^\lambda - c_1 g^{\alpha\delta}(\gamma^\sigma{}_{o\delta} + \\ + \tanh\beta \; c^\beta C^\sigma{}_{\beta\delta}) = 0. \tag{6.16b}$$

The fluid kinematic quantities are defined as

Accelaration

$$\dot{u}_a = u_{a;b}u^b. \tag{6.17a}$$

Expansion

$$\theta_{ab} = u_{(a;b)} - \dot{u}_{(a;b)}. \tag{6.17b}$$

Shear

$$\sigma_{ab} = \theta_{ab} - (1/3)\,\theta h_{ab}, \tag{6.17c}$$

where $\theta = \theta^a{}_a$ is the expansion scalar and $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor on the hypersurfaces S(t).

Vorticity

$$\omega_{ab} = u_{[a;b]} - \dot{u}_{[a}u_{b]}. \tag{6.17d}$$

From King and Ellis (1973) we can write the accelaration vector as

$$\dot{u}_a = (\gamma - 1)\theta \tanh\beta\,(\sinh\beta\, n_a + \cosh\beta\, c_a), \tag{6.18}$$

which shows immediately that an expanding tilted ($\beta \neq 0$) dust Universe model is always acceleration free. The vorticity vector is given by

$$\omega^a = (1/2)\sinh\beta\ \eta^{abcd}u_b c_{c;d}, \tag{6.19}$$

which shows that for a non-tilted ($\beta = 0$) fluid the vorticity vanishes. This is a well known result; the fluid 4-velocity becomes normal to the surfaces of homogeneity S(t) (Ellis, 1972).

The Einstein field equations are obtained when we substitute the corresponding left hand side of eqs. (6.10) by eqs. (6.7). If we choose the basis (e_a) such that eq. (5.13) is satisfied, the 6 geometrical variables $g_{\alpha\beta}(t)$ and the 4 matter variables $\mu(t)$, $u_\alpha(t)$ obey 14 ordinary differential equations (10 field equations plus 4 conservation equations). Of these 14 equations, 6 are second order in t and the rest are first order in t. The four eqs. (6.10b) and (6.10d) are first integrals of the other equations because of the conservation equations (6.16); so in general we may drop 4 of the equations (6.10c) involving second order derivatives in time and substitute them by the four first order ones (the second order will be automatically satisfied by virtue of the first order equations). Thus effectively we have to solve a system of 2 second order equations plus 8 first order ones. Of course care must be taken in the vacuum case and when $g_{\alpha\beta}$ are constants. In the first of these cases the conservation equations are trivially satisfied and the right hand side of eqs. (6.10) vanish, so essentially all the 10 equations must be taken into account. In the former case the equations become algebraic and the time-derivatives of the first order equations no longer imply validity of all the second order ones, so we can not drop the four second order equations anymore.

6.2 Blanchi type VII_h Universe

We are considering a basis (e_α) for which the structure functions $\gamma^\alpha{}_{\beta\lambda}$ turn out to be constants and chosen to assume the canonical values of table 1. The canonical form is achieved at a time t_0 by choosing the vectors (e_α) to be the eigenvalues of the symmetric pseudo-tensor $n^{\alpha\beta}$ and then rescaling their magnitudes to normalize the components of $n^{\alpha\beta}$. The eigenvectors are either necessarily orthogonal or they can be chosen so (in degenerate cases). Thus the ?

initial metric components $g_{\alpha\beta}(t_o)$ will be diagonal but not normalized,

$$g_{\alpha\beta}(t_o) = \mathrm{diag}(g_1, g_2, g_3). \tag{6.20}$$

The structure functions remain constant at later times when the basis is dragged along the normals n to the surface of homogeneity S(t). The field equations then determine the evolution of the metric components $g_{\alpha\beta}(t)$ from their initial values $g_{\alpha\beta}(t_o)$ (note that although these are diagonal initially, they will not in general be diagonal at later times, that is, for S(t), $t \neq t_o$). In this case the basis of vectors $\{e_\alpha\}$ has only spatial dependence, i. e. , $e_\alpha{}^\mu = e_\alpha{}^\mu(x^\sigma)$. However the automorphism group allows the basis $\{e_\alpha\}$ to be transformed without changing the canonically chosen structure constants. Therefore we can use these transformations to further simplify the metric at all times.

If we change the basis $\{e_\alpha\}$ by a <u>time-independent</u> automorphism transformation (see eqs. 5.1, 5.2, 5.4), the overall effect is to simplify the metric $g_{\alpha\beta}(t)$. The full time evolution of the new metric components $g_{\alpha'\beta'}(t)$ is still determined by the field equations. Nevertheless if a <u>time-dependent</u> automorphism group transformation is used some advantage arises because some of the metric time-dependence may be absorbed by the automorphism group parameters simplifying the new metric at all times without altering the structure constants $C^\alpha{}_{\beta\lambda}$: in particular we can diagonalize $g_{ab}(t)$ at all times in this way.

In fact the new metric assumes the form

$$ds^2 = -dt^2 + g_{\alpha'\beta'}(t)\Lambda^{\alpha'}{}_\alpha(t)\Lambda^{\beta'}{}_\beta(t)e^\alpha{}_\mu(x^\sigma)e^\beta{}_\nu(x^\lambda)\, dx^\mu\, dx^\nu, \tag{6.21}$$

where the 1-forms $w^\alpha = e^\alpha{}_\mu(x^\sigma)dx^\mu$ are the same invariant 1-form basis as before. We can use $\Lambda_{\alpha'}{}^\beta(t)$ to simplify $g_{\alpha\beta}(t)$, the penalty paid is that some of the $\gamma^\alpha{}_{0\beta}$ will be non-zero in the new basis $\{e_{\alpha'}\}$ (see eq. 5.4b). They vanish in the initial basis (e_α). These non-zero $\gamma^\alpha{}_{0\beta}$ are restricted by the equations (5.14).

To see how this simplification is gained, let us consider a particular example being the Bianchi type VII_h, our reference model.

The automorphism group of type VII_h can be written in terms of its 4-parameter family as (see table 2)

$$A = A_7 . A_1 . A_4, \tag{6.22a}$$

where the transformations A's are given in section 5.4. The transformation A takes the form

$$A_{\alpha'}{}^\beta = \begin{pmatrix} e^{-\lambda}\cos\alpha & e^{-\lambda}\sin\alpha & 0 \\ -e^{-\lambda}\sin\alpha & e^{-\lambda}\cos\alpha & 0 \\ -b_1 & -b_2 & 1 \end{pmatrix}, \tag{6.22b}$$

with λ, α and b_i $(i = 1, 2)$ functions of time.

Under (6.22b) the invariant basis (e_α) of type VII_h transforms to (see eq. 5.16)

$$\begin{aligned} e_{1'} &= e^{-\lambda}(\cos\alpha\, e_1 + \sin\alpha\, e_2), \\ e_{2'} &= e^{-\lambda}(-\sin\alpha\, e_1 + \cos\alpha\, e_2), \\ e_{3'} &= -b_1 e_1 - b_2 e_2 + e_3, \end{aligned} \tag{6.22c}$$

where (e_α) is the time-indendent type VII_h invariant basis with canonical structure constants. The 1-forms (w^α) dual to (e_α) are given explicitly by (Jantzen, 1979)

$$w^1 = e^{-ax^3}(\cos x^3\, dx^1 + \sin x^3\, dx^2),$$

$$w^2 = e^{-ax^3}(-\sin x^3\, dx^1 + \cos x^3\, dx^2), \tag{6.23}$$

$$w^3 = dx^3.$$

The transformed basis, by construction, has the same commutators as the old one

$$[e_{1'}, e_{2'}] = 0,$$

$$[e_{2'}, e_{3'}] = e_{1'} - ae_{2'}, \tag{6.24a}$$

$$[e_{3'}, e_{1'}] = ae_{1'} + e_{2'},$$

However now the commutators $[e_o, e_{\alpha'}]$ are given by

$$[e_{o'}, e_{1'}] = -\dot{\lambda} e_{1'} + k\dot{\alpha} e_{2'},$$

$$[e_{o'}, e_{2'}] = -\dot{\alpha} e_{1'} - \dot{\lambda} e_{2'}, \tag{6.24b}$$

$$[e_{o'}, e_{3'}] = -B_1 e_{1'} - B_2 e_{2'},$$

where k is a constant and

$$B_1(t) = (\dot{b}_1 \cos\alpha + \dot{b}_2 \sin\alpha)e^{-\lambda},$$

$$B_2(t) = (-\dot{b}_1 \sin\alpha + \dot{b}_2 \cos\alpha)e^{-\lambda}. \tag{6.24c}$$

It follows straightfowardly from eqs. (6.24b) that the quantities $\gamma^{\alpha'}{}_{o'\beta'}$ are

$$\gamma^{1'}{}_{o'1'} = \gamma^{2'}{}_{o'2'} = -\dot{\lambda}$$

$$\gamma^{1'}{}_{o'2'} = -\dot{\alpha}$$

$$\gamma^{1'}{}_{o'3'} = - B_1 \tag{6.25a}$$

$$\gamma^{2'}{}_{o'1'} = k\dot{\alpha}$$

$$\gamma^{2'}{}_{o'3'} = - B_2,$$

and the remaining non-zero commutators give

$$\gamma^{1'}{}_{2'3'} = C^{1'}{}_{2'3'} = 1, \tag{6.25b}$$

$$\gamma^{2'}{}_{1'3'} = C^{2'}{}_{1'3'} = - k, \tag{6.25c}$$

$$\gamma^{1'}{}_{1'3'} = C^{1'}{}_{1'3'} = \gamma^{2'}{}_{2'3'} = C^{2'}{}_{2'3'} = - a. \tag{6.25d}$$

The constant k has been introduced here to facilitate the discussion of other Bianchi types in the next section. It has a value of $k = 1$ for the Bianchi type VII_h.

The metric components $g_{\alpha\beta}(t) = e_\alpha \cdot e_\beta$ are in general arbitrary symmetric non-singular 3-dimensional matrices. By use of the automorphism group transformations (6.22b) and eqs. (5.2c), after a tedious calculation, some of the metric time-dependence can be absorbed in the automorphism parameters leading to the reduced form of $g_{\alpha'\beta'} = e_{\alpha'} \cdot e_{\beta'}$ as

$$g_{\alpha'\beta'} = \mathrm{diag}(e^{2A}, e^{-2A}, e^{2B}), \tag{6.26a}$$

and its inverse

$$g^{\alpha'\beta'} = \mathrm{diag}(e^{-2A}, e^{2A}, e^{-2B}), \tag{6.26b}$$

where $A = A(t)$, $B = B(t)$. To get the metric (6.26a) we have assumed that $g_{11}g_{22} - g_{12}{}^2 \neq 0$. The function $A(t)$ is such that $A(t) \neq 0$, for all t, to allow the fourth automorphism parameter to be used to reduce the metric to a quasi-Taub form T_3 where the component $g_{1'1'}$ is the inverse of the component $g_{2'2'}$ (Jantzen, 1979).

The Ricci tensor components are obtained from (6.7), (6.25) and (6.26). Explicitly they are

$$R_{oo} = -2\ddot{\lambda} - \ddot{B} - 2(\dot{A}^2 + \dot{\lambda}^2) - \dot{B}^2 - 1/2\,\dot{\alpha}^2(e^{2A} - ke^{-2A})^2 -$$

$$- 1/2\ B_1^{\,2}e^{2(A-B)} - 1/2\ B_2^{\,2}e^{-2(A+B)}, \tag{6.27a}$$

$$R_{o1} = 3/2\ aB_1e^{2(A-B)} - 1/2\ kB_2e^{-2(A+B)}, \tag{6.27b}$$

$$R_{o2} = -3/2\ aB_2e^{-2(A+B)} + 1/2\ B_1e^{2(A-B)}, \tag{6.27c}$$

$$R_{o3} = 2a(\dot{\lambda} - \dot{B}) - 1/2\ \dot{\alpha}(e^{2A} - ke^{-2A})^2, \tag{6.27d}$$

$$R_{11} = (\ddot{A} + \ddot{\lambda})e^{2A} + (\dot{A} + \dot{\lambda})(\dot{B} + 2\dot{\lambda})e^{2A} - 1/2\ B_1^{\,2}e^{4A-2B}$$

$$- 2a^2e^{2(A-B)} - 1/2\ (\dot{\alpha}^2 - e^{-2B})(e^{4A} - k^2e^{-4A})e^{2A}, \tag{6.27e}$$

$$R_{12} = [1/2\ \ddot{\alpha} + 1/2\ \dot{\alpha}(\dot{B} + 2\dot{\lambda}) + ae^{-2B}](e^{2A} - k^{-2A}) + 2\dot{\alpha}\dot{A}(e^{2A} + ke^{-2A})$$

$$- 1/2\ B_1B_2e^{-2B}, \tag{6.27f}$$

$$R_{13} = 1/2\ \dot{B}_1e^{2A} - 1/2\ kB_2\dot{\alpha}e^{-2A} + 1/2\ B_1(3\dot{\lambda} + 2\dot{A} - \dot{B})e^{2A}, \tag{6.27g}$$

$$R_{22} = (\ddot{\lambda} - \ddot{A})e^{-2A} + (\dot{\lambda} - \dot{A})(\dot{B} + 2\dot{\lambda})e^{-2A} - 1/2\ B_2^{\,2}e^{-2(2A+B)}$$

$$- 2a^2e^{-2(A+B)} + 1/2\ (\dot{\alpha}^2 - e^{-2B})(e^{4A} - k^2e^{-4A})e^{-2A}, \tag{6.27h}$$

$$R_{23} = 1/2\, \dot{B}_2 e^{-2A} + 1/2\, B_1 \dot{\alpha} e^{2A} + 1/2\, B_2 (3\dot{\lambda} - 2\dot{A} - \dot{B}) e^{-2A}, \tag{6.27i}$$

$$R_{33} = \ddot{B} e^{2B} + \dot{B}(\dot{B} + 2\dot{\lambda}) e^{2B} - 1/2\, (e^{2A} - ke^{-2A})^2 - 2a^2$$

$$+ 1/2\, B_1{}^2 e^{2A} + 1/2\, B_2{}^2 e^{-2A}. \tag{6.27j}$$

These imply the first order equation

$$R_{oo} + g^{\alpha\beta} R_{\alpha\beta} = R_{oo} + e^{-2A} R_{11} + e^{2A} R_{22} + e^{-2B} R_{33}$$

$$= 2\dot{\lambda}^2 - 2\dot{A}^2 + 4\dot{B}\dot{\lambda} - 6a^2 e^{-2B} - 1/2\, B_1{}^2 e^{2(A-B)}$$

$$- 1/2\, B_2{}^2 e^{-2(A+B)} - 1/2\, (\dot{\alpha}^2 + e^{-2B})(e^{2A} - ke^{-2A})^2. \tag{6.27k}$$

The conservation equations (6.16) now read

$$(w \cosh\beta\, e^{B+2\lambda})^{\cdot} - 2awc^3 \sinh\beta\, e^{B+2\lambda} = 0, \tag{6.28a}$$

$$(r \sinh\beta\, e^{2A+\lambda})^{\cdot} - re^{2A} \sinh\beta\, [ac^1 c^3 \tanh\beta +$$

$$+ kc^2 e^{-4A} (\dot{\alpha} + c^3 \tanh\beta)] = 0, \tag{6.28b}$$

$$(r \sinh\beta\, e^{-2A+\lambda} c^2)^{\cdot} + re^{-2A} \sinh\beta\, [-ac^2 c^3 \tanh\beta +$$

$$+ kc^1 e^{4A} (\dot{\alpha} + c^3 \tanh\beta)] = 0, \tag{6.28c}$$

$$(re^{2B} \sinh\beta\, c^3)^{\cdot} + r \sinh\beta\, \{c^1 e^{2A} [B_1 + \tanh\beta\, (ac^1 - c^2)] +$$

$$+ c^2 e^{-2A} [B_2 + \tanh\beta\, (ac^2 + kc^1)]\} = 0, \tag{6.28d}$$

where from (6.8b)

$$(c^1)^2e^{2A} + (c^2)^2e^{-2A} + (c^3)^2e^{2B} = 1. \qquad (6.28e)$$

We may consider equations (6.27b - k) as equations for the 6 geometric variables B_1, B_2, α, λ, A and B; and equations (6.28a-d) as equations for the matter variables μ, β and c^λ. One can explicitly check that in general the 4 equations (6.27b,c,d,k) are first integrals of the others (take the time derivative of these four equations; they are identically satisfied in view of (6.27e-j) and (6.28)). Thus we have to consider only 2 of the 6 equations (6.27e-j), the others will automatically be satisfied if all the remaining equations are true. So we can consider as the essential field equations to be solved the set $\{E_1\} \equiv \{$ (6.28a-d)$\}$ plus the set $\{E_2\} \equiv$ $\{$(6.27b-d,g,i,k),(6.11b-d)$\}$. Once these equations have been solved, (6.24c) must be solved for b_1 and b_2. Then eqs. (6.21), (6.22b), (6.23) and (6.26) enable us to find the explicit form of the metric (6.21).

Although this is a general procedure, exceptional cases arise when considering vacuum and constant geometric variables. These two cases must be treated carefully. In particular the integrability conditions must be checked and the set of equantions $\{E_3\} \equiv \{$(6.27e,f,h,j) and (6.10d)$\}$ must also be taken into account because some of them might be independent of the set $\{E_1, E_2\}$.

The set of field equations (6.27 plus 6.10) has been written in a synchronous time; that means the lapse function is $N = 1$. To achieve further simplification to the field equations we define a new time coordinate (non-synchronous gauge) t' along the world lines with a lapse function given by

$$N(t) = e^{-B}, \qquad (6.29a)$$

where the old time derivatives are given by

$$\dot{f} = N(t)\, f' = e^{-B} f', \tag{6.29b}$$

$$\ddot{f} = e^{-2B}(f'' - Bf'). \tag{6.29c}$$

where the prime (′) denotes the derivative with respect to the new time t′.

In the non-synchronous time t′ the equations (E_2) read

$$e^{2B}(\mu + p)\sinh\beta \cosh\beta\; c^1 = -3/2\; aB_1 - 1/2\; kB_2 e^{-4A}, \tag{6.30b}$$

$$e^{2B}(\mu + p)\sinh\beta \cosh\beta\; c^2 = -3/2\; aB_2 + 1/2\; B_1 e^{4A}, \tag{6.30c}$$

$$e^{3B}(\mu + p)\sinh\beta \cosh\beta\; c^3 = 2a(\lambda' - B') - 1/2\; \alpha'(e^{2A} - ke^{-2A})^2, \tag{6.30d}$$

$$e^{3B}(\mu + p)\sinh^2\beta\; c^1 c^3 = 1/2\; B_1' + 1/2\; B_1(3\lambda' + 2A' - B') - \\ - 1/2\; kB_2\alpha' e^{-4A}, \tag{6.30g}$$

$$e^{3B}(\mu + p)\sinh^2\beta\; c^2 c^3 = 1/2\; B_2' - 1/2\; B_2(2A' - 3\lambda' + B') + \\ + 1/2\; B_1\alpha' e^{4A}, \tag{6.30i}$$

$$2e^{2B}[\mu + (\mu + p)\sinh^2\beta\,] = 2\lambda'^2 - 2A'^2 + 4\lambda' B' - 6a^2 - \\ -1/2\; B_1^{\,2} e^{2A} - 1/2\; B_2^{\,2} e^{-2A} - 1/2\; (\alpha'^2 + 1)(e^{2A} - ke^{-2A})^2. \tag{6.30k}$$

The conservation equations (E_1) remains the same but with the changing of the time derivatives according to (6.29). The equantions (E_3) read

$$e^{2B}[1/2\; (\mu - p) + (\mu + p)\sinh^2\beta\; e^{2A}(c^1)^2] = A'' + \lambda'' + 2\lambda'(\lambda' + A') - \\ - 2a^2 - 1/2\; B_1^{\,2} e^{2A} - 1/2\; (\alpha'^2 - 1)(e^{4A} - k^2 e^{-4A}), \tag{6.30e}$$

$$e^{2B}[1/2\; (\mu - p) + (\mu + p)\sinh^2\beta\; e^{-2A}(c^2)^2] = \lambda'' - A'' + 2\lambda'(\lambda' - A') - \\ - 2a^2 - 1/2\; B_2^{\,2} e^{-2A} + 1/2\; (\alpha'^2 - 1)(e^{4A} - k^2 e^{-4A}), \tag{6.30h}$$

$$e^{2B}[1/2\ (\mu - p) + (\mu + p)\sinh^2\beta\ e^{2B}(c^3)^2] = B'' + 2B'\lambda' -$$

$$- 2a^2 + 1/2\ B_1{}^2 e^{2A} + 1/2\ B_2{}^2 e^{-2A} - 1/2\ (e^{2A} - ke^{-2A})^2, \qquad (6.30j)$$

$$e^{2B}(\mu + p)\sinh^2\beta\ c^1 c^2 = (1/2\ \alpha' + \alpha'\lambda' + a)(e^{2A} - ke^{-2A})$$

$$- 1/2\ B_1 B_2 + 2\alpha' A'(e^{2A} + ke^{-2A}). \qquad (6.30f)$$

To see the consequences of such simplifications, we will discuss in section 6.4 the vacuum, orthogonal and tilted cases. In the following section we present an extension of the results obtained for Bianchi type VII_h to some other Bianchi Universe models.

6.3 Extension to other Bianchi types

A similar procedure to the methods used in the previous section can also be applied in a straightforward manner to obtain simplified field equations to other Bianchi types. For each Bianchi type the metric may always be diagonalized (Jantzen, 1979) at all times which brings a great computational advantage due to the facility of finding its inverse matrix form.

For all class B Bianchi types, except type V, and for some of the class A types, one can obtain their field equations just by a simple interchange of variables in the equations already given for type VII_h (eqs. 6.30). This follows from the fact that all class B types, except type V, have a 4-dimensional automorphism group as well as the class A types VI_o and VII_o. The metric of all these types can be reduced at all times to the simple quasi-Taub form given in eq. (6.26). The structure functions are different in each case; however if we give appropriate values to the quantities k and a, and change the automorphism parameters according to the Bianchi type structure in consideration, an extension of the results of type VII_h to these other types may be given directly. This extension (which corrects that one in Siklos, 1980) is presented in table 3 below.

Table 3

Bianchi type extensions form type VII_h

VII_h		VI_h	IV	VII_o	VI_o	III
$k(=1)$	$\rightarrow$	-1	0	1	-1	-1
a	$\rightarrow$	a	1	0	0	1
α	$\rightarrow$	ϕ	b_3	α	ϕ	ϕ
B_1	$\rightarrow$	C_1	D_1	B_1	C_1	C_1
B_2	$\rightarrow$	C_2	D_2	B_2	C_2	C_2

where

$$C_1 = (\dot{b}_1 \cosh \phi + \dot{b}_2 \sinh \phi)e^{\lambda}, \tag{6.31a}$$

$$C_2 = (\dot{b}_1 \sinh \phi + \dot{b}_2 \cosh \phi)e^{\lambda}, \tag{6.31b}$$

$$D_1 = (\dot{b}_1 + b_3\dot{b}_2)e^{\lambda}, \tag{6.31c}$$

$$D_2 = \dot{b}_2 e^{\lambda}, \tag{6.31d}$$

are written according to the automorphism parameters defined in section 5.4.

Although the extension above holds to all class B types with the exception of type V, it deals only with the class A types VI_o and VII_o. In sections 6.5 and 6.6 we discuss the application of this method in simplifying the metric and field equations for Bianchi type V and to the other class A Bianchi types respectively.

6.4 The vacuum, orthogonal and tilted cases

The vacuum case

In the vacuum case we have $\mu = p = 0$. The conservation equations (E_1) are trivially satisfied and the left-hand side of the equations (E_2, E_3) all vanish. The resulting system of equations is gauge invariant in the sense that a constant space-like coordinate transformation in the automorphism variables does not affect the equations; as an example if we take the variable λ and change to λ + constant, the system will remain the same. Thus the quantities b_i $(i = 1, 2)$, λ and also B do not occur explicitly in these equations; in this case only their first derivatives occur. We may regard equations (E_2) as algebraic equations for the variables B_a $(a = 3, 4, 5)$ defined by

$$B_3 = \alpha', \tag{6.32a}$$

$$B_4 = \lambda', \tag{6.32b}$$

$$B_5 = B', \tag{6.32c}$$

in addition to the first order equation for the variable A. Solving these equations, the dynamical variables b_1, b_2, α, λ and B are determined by the quadratures $(E_4) \equiv$ (eqs.(6.24c), (6.32)).

The argument above does not depend on the detailed form of the field equations. However regarding eqs. (6.30b,c) as a pair of algebraic equations for B_1 and B_2, the left-hand side is always non-zero, unless when

$$k = - 9a^2 \neq 0, \tag{6.33}$$

which corresponds to the exceptional case of Bianchi type VI_h with $h = -1/9$ (see table of 3). Therefore, excluding the case of type $VI_{h=-1/9}$ we must have

$$B_1 = 0 = B_2. \tag{6.34}$$

From equations (6.24) and (6.34) we have that

$$\dot{b}_1 = 0 \quad \Leftrightarrow \quad b_1 = \text{const.} \tag{6.35a}$$

$$\dot{b}_2 = 0 \quad \Leftrightarrow \quad b_2 = \text{const.} \tag{6.35b}$$

By virtue of equation (6.34), the equations (6.30g,i) are also trivially satisfied. This represents a degenerate case in which equations (E_3) are no longer automatically satisfied by virtue of (E_1) and (E_2).

The essential set of field equations may be chosen to be

$$a(B_4 - B_5) - 1/4\, B_3(e^{2A} - ke^{-2A})^2 = 0, \tag{6.36a}$$

$$A'^2 - B_4(B_4 + 2B_5) + 1/4\,(1 + B_3^2)(e^{2A} - ke^{-2A})^2 + 3a^2 = 0, \tag{6.36b}$$

$$B_4' + 2B_4^2 - 2a^2 = 0, \tag{6.36c}$$

$$B_5' + 2B_4B_5 - 1/2\,(e^{2A} - ke^{-2A})^2 - 2a^2, \tag{6.36d}$$

where the eq. (6.36c) has been obtained by adding eqs. (6.30e) and (6.30h).

For the class A types covered in table 3, we see from eq. (6.35a) that

$$B_3 = 0 \ \Rightarrow\ \alpha' = 0 \ \Leftrightarrow\ \alpha = \text{const.} \tag{6.37}$$

The result given in eqs. (6.35a,b) and (6.37) shows that for the class A types VI_o and VII_o the vacuum solutions must have the special-automorphism parameters, b_1, b_2 and α (or Φ for type VI_o) constant.

The orthogonal case

The orthogonal case has the energy-density

$$\mu \neq 0, \tag{6.39a}$$

and the fluid flow lines are orthogonal to the surfaces of homogeneity S(t),

$$u^a = n^a \iff \beta = 0. \tag{6.39b}$$

The three conservation equations (6.28b,c,d) are individually satisfied and the fourth one, (6.28a), can be integrated directly to give

$$w = w_o\, e^{-(B+2\lambda)}, \quad w_o = \text{const.} \neq 0. \tag{6.40}$$

The left-hand side of the equations (6.30b,c,d,g,i) still remain zero, then the same argument applied to the vacuum case leading to eq. (6.34) remains true. For the orthogonal case the automorphism parameters b_1 and b_2 must also be constant. The non-trivial equations are now (6.30d,k) plus two of the equations given by (E_3) in addition to eq. (6.40). In this case we see from (6.40) and (6.15e) that the variables B and λ <u>do</u> occur explicitly in the set of field equations, so we no longer can achieve a significant simplification by replacing them by B_4 and B_5 as done for the vacuum case (eqs. 6.32b,c). The equations are now essentially second order in t.

In the orthogonal case the accelaration $\dot{u}_a$ and the vorticity ω_{ab} vanish. The expansion scalar θ is given by

$$\theta = \dot{B} + 2\dot{\lambda}, \tag{6.41a}$$

and the shear tensor by

$$\sigma_{\alpha\beta} = \begin{bmatrix} 1/3\,(3\dot{A} - \dot{B} + \dot{\lambda})e^{2A} & 1/2\,\dot{\alpha}(e^{2A} - ke^{-2A}) & 1/2\,B_1e^{2A} \\ 1/2\,\dot{\alpha}(e^{2A} - ke^{-2A}) & 1/3\,(\dot{\lambda} - \dot{B} - 3\dot{A})e^{-2A} & 1/2\,B_2e^{-2A} \\ 1/2\,B_1e^{2A} & 1/2\,B_2e^{-2A} & 2/3\,(\dot{B} - 2\dot{\lambda})e^{2B} \end{bmatrix} \qquad (6.41b)$$

The open Friedmann-Robertson-Walker ($K = -1$) Universe is obtained by setting $\sigma_{\alpha\beta} = 0$, which gives

$$\dot{\alpha} = \dot{A} = 0, \qquad (6.42a)$$

$$\dot{B} = \dot{\lambda}, \qquad (6.42b)$$

$$B_1 = 0 = B_2. \qquad (6.42c)$$

By virtue of the eqs. (6.42) the equation (6.30d) is identically satisfied, and the remainig field equation is (6.30k) which might be expressed in the form

$$2\mu e^{2B} = 6B' + 6E, \qquad (6.43a)$$

where $E = -1/6 \sinh^2 2A - a^2 = \text{const.} < 0$. If we define a new function $R(t)$ by

$$R(t) = e^B, \qquad (6.43b)$$

and making use of eq. (6.29) the equation (6.43a) may be written in the form

$$(\dot{R}/R)^2 + E/R^2 = 1/3\,\mu, \qquad (6.43c)$$

which is the <u>Friedmann</u> <u>equation</u> (Ellis, 1972).

The tilted case

In the tilted case both the energy-density μ and the tilt angle β must be non-zero. The conservation equations (6.28) are all non-trivial and the left-hand side of the field equations will in general be non-zero. Although the variables chosen have simplified the field equations somewhat, in this case (which is the general situation) clearly the system is much more complex than for both the vacuum and the orthogonal system. Despite the complexity of the tilted system, we can still find a subsystem in which the equations (6.34) are valid; then form eq. (6.28e) we have

$$c^3 = e^{-B} \quad \text{and} \quad c^1 = c^2 = 0. \tag{6.44a}$$

From equations (6.28e) and (6.44a) we have

$$r\sinh\beta = r_0 e^{-B}, \quad r_0 = \text{const.} \neq 0, \tag{6.44b}$$

which characterizes the simplest subsystem that allows tilt. The essential field equations for this case are

$$(we^{B+2\lambda}\cosh\beta)' = 2awe^{2(B+\lambda)}\sinh\beta\, c^3, \tag{6.45a}$$

$$e^{2B}(\mu + p)\sinh\beta\cosh\beta = 2a(\lambda' - B') - 1/2\, B_3(e^{2A} - ke^{-2A})^2, \tag{6.45b}$$

$$2e^{2B}[\mu + (\mu + p)\sinh^2\beta] = 2\lambda'^2 - 2A'^2 + 4B'\lambda' - 6a^2 -$$

$$- 1/2\,(1 + B_3{}^2)(e^{2A} - ke^{-2A})^2, \tag{6.45e}$$

$$(\mu + p)e^{2B} = 2\lambda'' + 4\lambda'^2 - 4a^2, \tag{6.45f}$$

$$e^{2B}[1/2\,(\mu - p) + (\mu + p)\sinh^2\beta] = B'' + 2B'\lambda' - 2a^2 -$$

$$- 1/2\,(e^{2A} - ke^{-2A})^2, \tag{6.45g}$$

$$[B_3' + 2B_3\lambda' + 2a](e^{2A} - ke^{-2A}) + 4B_3A'(e^{2A} + ke^{-2A}) = 0, \qquad (6.45h)$$

where eq. (6.45f) has been obtained by adding eqs. (6.30e) and (6.30h).

6.5 The Bianchi Type V Universe

The Bianchi type V Lie algebra is invariant under the 6-parameter automorphism group of transformations given in table 2. The 4-parameter automorphism families

$$\Lambda_1 = (A_1, A_4, A_7),$$

$$\Lambda_2 = (A_3, A_4, A_7),$$

can be used to reduce the type V metric to the quasi-Taub form given in eq. (6.26). If we choose Λ_1 as the invariance family for type V, the equations (6.22) are still valid. This leads to eqs. (6.25a, d) with the values of $k = a = 1$. However now eqs. (6.25b,c) are no longer valid because in type V these structure constants vanish (see eq. 4.24). Therefore a simple extension from type VII_h field equations as suggested by Siklos (1980) can not be done to obtain the field equations for type V. Nevertheless as long as we know the metric and the structure functions and constants, the field equations may be calculated in a straightforward manner.

In this section we explore the freedom available to use an extra automorphism parameter in the invariance family of type V to futher simplify the metric compared with the form given by (6.26a). For this goal we choose the invariance family of type V to be given by

$$\Lambda = A_7.A_{2'}.A_4.A_3, \qquad (6.46a)$$

with A's given in section 5.4. The transformation Λ is given in terms of the automorphism parameters by

$$\Lambda_\alpha{}^\beta = \begin{pmatrix} e^{-(\phi+\lambda)} & 0 & 0 \\ -b_3 e^{\phi-\lambda} & e^{\phi-\lambda} & 0 \\ b_2 b_3 - b_1 & -b_2 & 1 \end{pmatrix} \tag{6.46b}$$

Following the same procedure developed for type VII_h the invariant basis (e_α) changes to

$$e_{1'} = e^{-(\phi+\lambda)}\, e_1, \tag{6.47a}$$

$$e_{2'} = e^{-(\phi+\lambda)}(-b_3\, e_1 + e_2), \tag{6.47b}$$

$$e_{3'} = (b_2 b_3 - b_1)\, e_1 - b_2\, e_2 + e_3. \tag{6.47c}$$

The metric $g_{\alpha'\beta'} = e_{\alpha'} \cdot e_{\beta'}$ may be reduced by use of equations (6.46a) and (5.2c) to the form

$$g_{\alpha'\beta'} = \mathrm{diag}(e^{2A}, {}^{-2A}, e^{-2A}), \tag{6.48}$$

where $A = A(t)$. We require that $A(t) \neq 0$, for all t, and that $g_{11}g_{22} - g_{12}{}^2 \neq 0$.

The commutators of the transformed basis $(e_{\alpha'})$ give the structure functions

$$\gamma^{1'}{}_{o'1'} = -(\dot{\lambda} + \dot{\phi}),$$

$$\gamma^{1'}{}_{o'2'} = -\dot{b}_3 e^{2\phi},$$

$$\gamma^{1'}{}_{o'3'} = -M_1 e^{\phi}, \tag{6.49a}$$

$$\gamma^{2'}{}_{o'2'} = -\dot{\lambda} + \dot{\phi},$$

$$\gamma^{2'}{}_{o'3'} = -M_2 e^{-\phi},$$

where the quantities $M_1(t)$ and $M_2(t)$ are defined as

$$M_1(t) = (\dot{b}_1 - b_2\dot{b}_3)e^{\lambda},$$

$$M_2(t) = \dot{b}_2 e^{\lambda}.$$

From eq. (4.24) and table 1 we have that

$$C^{1'}{}_{1'3'} = C^{2'}{}_{2'3'} = 1. \tag{6.49b}$$

The Ricci tensor components are then given by

$$R_{oo} = \ddot{A} - 2\ddot{\lambda} - 2(\dot{\lambda}^2 + \dot{\phi}^2) - 3\dot{A}^2 - 4\dot{A}\dot{\phi} - 1/2\ \dot{b}_3^{\,2} e^{4A+4\phi} - $$
$$- 1/2\ M_1^{\,2} e^{4A+2\phi} - 1/2\ M_2^{\,2} e^{-2\phi}, \tag{6.50a}$$

$$R_{o1} = -3/2\ M_1 e^{4A+\phi}, \tag{6.50b}$$

$$R_{o2} = -3/2\ M_2 e^{-\phi}, \tag{6.50c}$$

$$R_{o3} = 2(\dot{A} + \dot{\lambda}), \tag{6.50d}$$

$$R_{11} = (\ddot{A} + \ddot{\lambda} + \ddot{\phi})e^{2A} + (2\dot{\lambda} - \dot{A})(\dot{\lambda} + \dot{\phi} + \dot{A})e^{2A} - 1/2\ \dot{b}_3^{\,2} e^{6A+4\phi} - $$
$$- 1/2\ M_1^{\,2} e^{6A+2\phi} - 2e^{4A}, \tag{6.50e}$$

$$R_{12} = 1/2\ \ddot{b}_3 e^{2(A+\phi)} + 1/2\ (3\dot{A} + 3\dot{\phi} + 2\dot{\lambda})e^{2A} + 1/2\ M_1 M_2 e^{2A}, \tag{6.50f}$$

$$R_{13} = 1/2\ \dot{M}_1 e^{2A+\phi} + 1/2\ M_1(3\dot{A} + 3\dot{\lambda} + 2\dot{\phi})e^{2A+\phi}, \tag{6.50g}$$

$$R_{22} = (\ddot{\lambda} - \ddot{\phi} - \ddot{A})e^{-2A} + (2\dot{\lambda} - \dot{A})(\dot{\lambda} - \dot{\phi} - \dot{A})e^{-2A} + 1/2\ \dot{b}_3{}^2 e^{4\phi+2A} +$$

$$+ 1/2\ M_2{}^2 e^{-2(A+\phi)} - 2, \qquad (6.50h)$$

$$R_{23} = 1/2\ \dot{M}_2 e^{-(2A+\phi)} + 1/2\ M_2(3\dot{\lambda} - \dot{A} - 2\dot{\phi})e^{-(2A+\phi)} + 1/2\ M_1 b_3 e^{2A+3\phi}, \qquad (6.50i)$$

$$R_{33} = -\ddot{A}\, e^{-2A} - (2\dot{\lambda} - \dot{A})\dot{A}e^{-2A} + 1/2\ M_1{}^2 e^{2(A+\phi)} +$$

$$+ 1/2\ M_2{}^2 e^{-2(A+\phi)} - 2, \qquad (6.50j)$$

The first order equation (6.7e) is given by

$$R_{oo} + g^{\alpha\beta}R_{\alpha\beta} = 2\dot{\lambda}^2 - 2\dot{A}^2 - 2\dot{\phi}^2 - 4\dot{A}(\dot{\phi} + \dot{\lambda}) - 1/2\ \dot{b}_3{}^2 e^{4(A+\phi)} -$$

$$- 1/2\ M_1{}^2 e^{2(2A+\phi)} - 1/2\ M_2{}^2 e^{-2\phi} - 6e^{2A}, \qquad (6.591)$$

The conservation equations (6.10) are given by

$$(we^{2\lambda-A}\cosh\beta)^{\cdot} - 2we^{2\lambda-A}\sinh\beta c^3 = 0, \qquad (6.51a)$$

$$(re^{2A+\lambda-\phi}\sinh\beta c^1)^{\cdot} - r\sinh\beta\tanh\beta e^{2A+\lambda-\phi}c^1c^3 = 0, \qquad (6.51b)$$

$$(re^{\lambda-\phi-2A}\sinh\beta c^2)^{\cdot} - r\sinh\beta(c^2c^3\tanh\beta - \dot{b}_3 e^{4A-2\phi}c^1)e^{\lambda-\phi-2A} = 0, \qquad (6.51c)$$

$$(re^{-2A}\sinh\beta c^3)^{\cdot} + r\sinh\beta[c^1e^{2A}(M_1e^{\phi} + \tanh\beta c^1) + c^2e^{-2A}(M_2e^{-\phi} +$$

$$+ \tanh\beta c^2)] = 0. \qquad (6.51d)$$

The special automorphism parameters enter in the field equations only as first and second derivatives, so we may define a new function $B_3(t)$ by

$$B_3 = \dot{b}_3. \qquad (6.52)$$

For the vacuum case we have the conservation equations identically satisfied. Thus the automorphism parameter λ enter in the field equations only as first and second derivatives. This allows the change of variable

$$\dot{\lambda} = B_4. \tag{6.53}$$

The field equations (6.50b,c,d) give, respectively

$$M_1 = 0 \;\Rightarrow\; \dot{b}_1 = b_2\dot{b}_3, \tag{6.54a}$$

$$M_2 = 0 \;\Rightarrow\; \dot{b}_2 = 0 \;\Leftrightarrow\; b_2 = \text{const.}, \tag{6.54b}$$

$$\dot{A} - \dot{\lambda} = 0 \;\Leftrightarrow\; \dot{\lambda} = -\dot{A}. \tag{6.54c}$$

From eqs. (6.50f) and (6.54a,b) we have that

$$\dot{B}_3 + B_3(\dot{A} + 3\dot{\phi}) = 0, \tag{6.55a}$$

which does allows a solution with

$$B_3 = 0 \;\Rightarrow\; \dot{b}_3 = 0 \;\Leftrightarrow\; b_3 = \text{const.} \tag{6.55b}$$

Therefore, for type V vacuum we have a subsystem in which the 3 special automorphism parameters b_i $(i=1,2,3)$ are constants. Of course if condition (6.55b) does not hold, only the special automorphism parameter b_2 must be constant.

In the orthogonal case we may still carry out the change of variables given by eq. (6.53). Now the conservation equation (6.51a) has an explicit dependence on λ. Actually this equation can be integrated on the spot resulting in

$$w = w_o e^{A-2\lambda}, \quad w_o = \text{const.} \neq 0. \tag{6.56}$$

Thus we can no longer assume the change of variable (6.53). In this case as well we may have a subsystem in which the 3 special automorphism parameters b_i

$(i=1,2,3)$ are constant.

Although the general tilted case is rather complicated, the simplest subsystem which allows tilt is obtained when the eqs. (6.54a,b) are still valid. In this case we have

$$c^1 = c^2 = 0, \quad c^3 = e^A, \tag{6.57a}$$

and from the conservation equation (6.51d)

$$r\sinh\beta = r_o e^A, \quad r_o = \text{const.} \neq 0. \tag{6.57b}$$

6.6 The Class A Bianchi Types

The class A Bianchi types VI_o and VII_o have been considered as extensions from type VII_h results as shown in table 2. The other class A types - I, II, VIII and IX - can also have their metric and field equations simplified by means of the automorphism group. The metric of types VIII and IX may at most be reduced to a diagonal form (Jantzen, 1979) as they have a 3-dimensional automorphism group. The Bianchi type I metric is already very simple so that application of this method does not bring any significant advantage. Type I has its metric in a diagonal form in a suitable coordinate frame (Ellis and MacCallum, 1969).

a diagonal form depends on what stress-energy tensor form is used (perfect fluid allows a diagonal form; E-M does not necessarily).

The most interesting case is then type II which has a 6-dimensional automorphism group. From table 2 we can choose a 5-parameter invariance family for type II given by

$$\Lambda = A_{2'}.A_3.A_1.A_5.A_6, \tag{6.58a}$$

where the A's are given in section 5.4.

The transformation (6.58a) is written in terms of the automorphism group parameters as

$$\Lambda_\alpha{}^\beta = \begin{pmatrix} e^{\rho-\phi}\cos\alpha & e^{\rho-\phi}\sin\alpha & -N_1 e^{\rho-\phi} \\ -e^{\rho+\phi}\sin\alpha & e^{\rho+\phi}\cos\alpha & -N_2 e^{\rho+\phi} \\ 0 & 0 & e^{2\rho} \end{pmatrix} \tag{6.58b}$$

where

$$N_1 = b_5\cos\alpha + b_6\sin\alpha,$$

$$N_2 = -b_5\sin\alpha + b_6\cos\alpha. \tag{6.58c}$$

The transformed invariant basis $\{e_{\alpha'}\}$ is given in terms of the old one by

$$e_{1'} = e^{\rho-\phi}(\cos\alpha\ e_1 + \sin\alpha\ e_2 - N_1\ e_3), \tag{6.59a}$$

$$e_{2'} = e^{\rho+\phi}(-\sin\alpha\ e_1 + \cos\alpha\ e_2 - N_2\ e_3), \tag{6.59b}$$

$$e_{3'} = e^{2\rho}\ e_3. \tag{6.59c}$$

The structure functions obtained from the commutators of the new basis are

$$\gamma^{1'}{}_{o'1'} = \dot{\rho} - \dot{\phi}$$

$$\gamma^{1'}{}_{o'2'} = -\dot{\alpha}e^{2\phi}$$

$$\gamma^{2'}{}_{o'1'} = \dot{\alpha}e^{-2\phi}$$

$$\gamma^{2'}{}_{o'2'} = \dot{\rho} + \dot{\phi}$$

$$\gamma^{3'}{}_{o'1'} = -H_1 e^{-\phi} \tag{6.60a}$$

$$\gamma^{3'}{}_{o'2'} = -H_2 e^{\phi}$$

$$\gamma^{3'}{}_{o'3'} = 2\dot{\rho},$$

where

$$H_1(t) = (\dot{b}_5 \cos\alpha + \dot{b}_6 \sin\alpha) e^{-\rho},$$

$$H_2(t) = (-\dot{b}_5 \sin\alpha + \dot{b}_6 \cos\alpha) e^{-\rho}. \tag{6.60b}$$

From eq. (4.24) the non-zero type II structure constant is given by

$$C^{3'}{}_{1'2'} = 1. \tag{6.60c}$$

The metric $g_{\alpha'\beta'} = e_{\alpha'} \cdot e_{\beta'}$ can be reduced to the form

$$g_{\alpha'\beta'} = \mathrm{diag}(\, e^{2A},\ e^{2A},\ e^{-4A}), \tag{6.61}$$

with $A = A(t)$. We require that $A(t) \neq 0$ for all t, and that $g_{23} - g_{22} \neq 0$.

The Ricci tensor components turn out to be

$$R_{oo} = 4\ddot{\rho} - 6\dot{\rho}^2 - 6\dot{A}^2 - 2\dot{\phi}^2 - 4\dot{A}\dot{\rho} + \dot{\alpha}^2(1 - \cosh 4\phi) - 1/2\ H_1{}^2 e^{-2(3A-\phi)} -$$
$$- \ 1/2\ H_2{}^2 e^{-2(3A-\phi)}, \tag{6.62a}$$

$$R_{o1} = 1/2\ H_2 e^{-6A+\phi}, \tag{6.62b}$$

$$R_{o2} = 1/2\ H_1 e^{-6A-\phi}, \tag{6.62c}$$

$$R_{o3} = 0, \tag{6.62d}$$

$$R_{11} = (\ddot{A} + \ddot{\phi} - \ddot{\rho})e^{2A} + 4\dot{\rho}(\dot{\rho} - \dot{\phi} - \dot{A})e^{2A} - \dot{\alpha}^2 e^{2A} \sinh 4\phi +$$

$$+ 1/2\, H_1^2 e^{-4A-2\phi} - 1/2\, e^{-6A}, \tag{6.62e}$$

$$R_{12} = \ddot{\alpha} \cosh 2\phi e^{2A} + 4\alpha\dot{\phi} e^{2A} \cosh 2\phi - 4\alpha\dot{\rho} e^{2A} \sinh 2\phi + 1/2\, H_1 H_2 e^{-4A}, \tag{6.62f}$$

$$R_{13} = 1/2\, \dot{H}_1 e^{-4A-\phi} - 1/2\, H_1(2\dot{\phi} + 6\dot{A} + 5\dot{\rho}) e^{-4A-\phi} - 1/2\, H_2 \dot{\alpha} e^{-4A+3\phi}, \tag{6.62g}$$

$$R_{22} = (\ddot{A} - \ddot{\rho} - \ddot{\phi}) e^{2A} + 4\dot{\rho}(\dot{\rho} + \dot{\phi} - \dot{A}) e^{2A} + \dot{\alpha}^2 e^{2A} \cosh 4\phi +$$

$$+ 1/2\, H_2^2 e^{-4A+2\phi} - 1/2\, e^{-6A}, \tag{6.62h}$$

$$R_{23} = 1/2\, \dot{H}_2 e^{-4A+\phi} + 1/2\, H_2(2\dot{\phi} - 6\dot{A} - 5\dot{\rho}) e^{-4A+\phi} + 1/2\, H_1 \dot{\alpha} e^{-4A-3\phi}, \tag{6.62i}$$

$$R_{33} = -2(\ddot{A} + \ddot{\rho}) e^{-4A} + 8\dot{\rho}(\dot{\rho} + \dot{A}) e^{-4A} - 1/2\, H_1^2 e^{-10A-2\phi} -$$

$$- 1/2\, H_2^2 e^{-10A+2\phi} + 1/2\, e^{-10A}, \tag{6.62j}$$

These imply the first order equation

$$R_{oo} + g^{\alpha\beta} R_{\alpha\beta} = 10\dot{\rho}^2 - 6\dot{A}^2 - 2\dot{\phi}^2 - 4\dot{A}\dot{\rho} + \dot{\alpha}^2(1 - \cosh 4\phi) - 1/2\, H_1^2 e^{-6A-2\phi} -$$

$$- 1/2\, H_2^2 e^{-6A+2\phi} - 1/2\, e^{-8A}. \tag{6.62k}$$

The field equations are now easily obtained by use of (6.62) and (6.10). The conservation equations turn out to be

$$(we^{-4\rho} \cosh\beta)^{\cdot} = 0, \tag{6.63a}$$

$$(re^{-2A-\phi+\rho} c^1 \sinh\beta)^{\cdot} + r\sinh\beta e^{-2A-\phi+\rho} [\dot{\alpha} e^{-2\phi} c^2 + (\tanh\beta c^2 -$$

$$- H_1 e^{-\phi}) e^{-6A} c^3], \tag{6.63b}$$

$$(re^{2A-\phi-\rho} \sinh\beta c^2)^{\cdot} + r\sinh\beta e^{-2A+\phi+\rho} [\dot{\alpha} e^{2\phi} c^1 + (H_2^{2\phi} -$$

$$- \tanh\beta c^1) e^{-6A} c^3], \tag{6.63c}$$

$$(re^{-4A-2\phi} \sinh\beta c^3)^{\cdot} = 0. \tag{6.63d}$$

The conservation equations (6.63a,d) can be immediately integrated given, respectively

$$w\cosh\beta = w_o e^{4\rho}, \quad w_o = \text{const.} \neq 0, \tag{6.64a}$$

$$r\sinh\beta = r_o e^{4A+2\rho}, \quad r_o = \text{const.} \neq 0. \tag{6.64b}$$

In the vacuum case the conservation equations are identically satisfied. The field equations give

$$H_1 = 0, \tag{6.64a}$$

$$H_2 = 0. \tag{6.64b}$$

From eqs. (6.64) and (6.60b) we have that

$$\dot{b}_5 = 0 \quad \Longleftrightarrow \quad b_5 = \text{const.} \tag{6.65a}$$

$$\dot{b}_6 = 0 \quad \Longleftrightarrow \quad b_6 = \text{const.} \tag{6.65b}$$

The automorphism parameters ρ and α do not occur explicitly in the field equations, only their first and second derivatives. Then we may define new variables H_3 and H_4 by

$$H_3 = \dot{\rho}, \tag{6.66a}$$

$$H_4 = \dot{\alpha}. \tag{6.66b}$$

The remaining set of field equations turns out to be second order only in the variable A. From eq. (6.56f) we have

$$\dot{H}_4\cosh 2\Phi + 4H_4(\dot{\Phi}\cosh 2\Phi - \dot{\rho}\sinh 2\Phi) = 0, \tag{6.66a}$$

which allows a solution with $H_4 = 0$, resulting in α = constant. Therefore in the vacuum case we must have the 2 special automorphism parameters b_5 and b_6 constant, and there exists a subsystem in which the special automorphism parameter α is also constant.

In the orthogonal case the only non-trivially satisfied conservation equation is (6.62e). Thus we can still have eqs. (6.64) and (6.66b) valid. However we can no longer make use of eq. (6.66a) due to the conservation equation (6.63e). In this case we must also have the special automorphism parameters b_5 and b_6 constant, and there is a subsystem in which the special automorphism parameter α is constant.

The tilted case is much more complicated. Nevertheless there is a subsystem in which the eqs. (6.65) remain true. This leads to

$$r \sinh\beta = r_o e^{2(A+\beta)}, \tag{6.67a}$$

which was obtained form eq. (6.63f) and the fact that

$$c^1 = c^2 = 0, \quad c^3 = e^{2A}. \tag{6.67b}$$

This characterizes the simplest subsystem that allows tilt. As has been shown by King and Ellis (1973) type II may have tilt but it is vorticity free.

6.7 Concluding remarks

In Chapter 5 we have presented the automorphism group and the special automorphism group for each Bianchi group type and written them as individual transformations with 1- or 2-parameters. The invariance family for each Bianchi group type is given in Table 2, where the automorphism group dimension and the group orbit dimension are also displayed.

In this Chapter we have shown how the time-dependent automorphism group of the Bianchi Lie algebras can be used to diagonalize or further simplify the metric of the Bianchi Universes. The simplification in the metric is achieved when some of the metric time-dependence is absorbed in some of the automorphism group parameter. For the semisimple Bianchi types (VIII and XI) the metric may at most be diagonalized (Jantzen, 1979) due to the fact that these models have a 3-dimensional automorphism group (for the semisimple Bianchi types the automorphism group coincides with the special automorphism group). The non-abelian non-semisimple Bianchi types (II, IV, V, VI_h and VII_h) have a larger automorphism group which provides extra freedom to further simplify the metric. In fact, the non-abelian non-semisimple Bianchi types IV, VI_h and VII_h have a 4-parameter automorphism group which allows us to reduce the metric to a quasi-Taub form T_3, that is to a two parameter diagonal form in which one of the metric components is just the inverse of the other. The other non-abelian non-semisimple Bianchi types have a 6-dimensional automorphism group; thus a further simplification may be possible using one extra automorphism parameter to reduce the metric to a 1-parameter diagonal form. The abelian Bianchi type I has a 9-parameter automorphism group, however its metric can be written in a diagonal form in a coordinate frame. This exhibits a great simplicity without the necessity of appeal to the automorphism group in order to diagonalize, although

see note p. 86.

a further simplification might by possible.

Thus all Bianchi Universes models can be written in an 'automorphism basis' where the metric is at least in a diagonal form. The price that must be paid for this is that, although the automorphism basis is still an invariant basis, it is no longer invariant by dragging along the normals. This generates some time-dependent structure functions in addition to the structure constants due to the non-commutativity of the automorphism vector basis with the normal vector n = $\mathbf{e}_0$.

The application of the automorphism group to reduce the 3-metric time-dependence is considered explicitly in section 6.2 for Bianchi type VII_h Universes. The Einstein field equations and the conservation equations are also written down for type VII_h. However, by a simple change of variables of type VII_h, the field equations and the conservation equations are immediately obtained for some other Bianchi types. This is explained in section 6.3, where Table 3 shows which and how type VII_h quantities should be changed in order to get the extension. Our results correct and extend the previous work of Siklos (1980).

Siklos (1980) has transformed the automorphism basis to an orthonormal frame. However, as we have shown, this is not necessary in order to see how the metric can be simplified via the automorphism group. The automorphism basis given in eqs. (6.22c) with metric (6.26a), can be easily transformed to an orthonormal frame. In this orthonormal frame, the vector a_λ and the matrix $n^{\alpha\beta}$ are no longer constant; they are transformed to $A_\lambda = (0,\ 0,\ ae^{-B})$ and $n^{\alpha\beta} = \mathrm{diag}(e^{2A-B},\ ke^{-(2A+B)},\ 0)$. These expressions also correct the previous work of Siklos by an appropriate identification of the parameters: (his a^2 is identified with our e^{-2B} and his $e^{2\mu}$ with our e^{2A}).

In a recent series of papers Jantzen (1979, 1983, 1984) has also made use of the automorphism group together with the ADM Hamiltonian formalism to investigate the dynamics of the spatially homogeneous perfect fluid universe models. In his approach the Einstein field equations are given by a set of 12 first order ordinary differential equations plus four constraints. The constraints equations can be used to replace all matter variables by geometric variables, and the field equations become a 12-dimensional autonomous system. Following the Hamiltonian formulation of Jantzen and using the scale invariance of the Einstein field equations (Jantzen, 1983) and the compactification of the gravitational phase space, Rosquist (1983, 1984) has found a new exact Bianchi type VI_0 cosmological solution which represents a tilted, rotating and expanding radiation filled Universe. The solution is an equilibrium solution of a 4-dimensional autonomous subsystem of type VI_0. In a similar procedure, the compactified field equations for class B Bianchi types have been obtained by Roque and Jaklitsh (1985); in particular a 1-parameter families of vacuum solutions have been found for Bianchi types IV, VI_h and VII_h Universes (see preprint copy in the Appendix).

The geometric approach presented in this Chapter may also be suitably transformed to give the Einstein field equations as a set of 12 first order ordinary differential equations indentical to those provided by the Hamiltonian formalism. However, for this aim the use of only three special automorphism group parameters in order to diagonalize the metric of the non-abelian non-semisimple Bianchi types have shown to be more appropriate than the use of extra automorphism group parameters for further simplification of the metric because they generate undesirable structure functions to make the appropriate identifications. A comparative analysis between the geometric approach presented here and the Hamiltonian formalism of Jantzen is currently being investigated.

For the vacuum Bianchi Universes presented in Table 3, a great simplification of the Einstein field equations is possible through the use of the time-dependent automorphism group. The vacuum and orthogonal cases for Bianchi types VI_0 and VII_0 must have constant special automorphism group parameters, while the Bianchi types IV, VI_h and VII_h only two of the special automorphism group parameters are required to be constant. This simply means that for the vacuum and orthogonal Bianchi types VI_0 and VII_0, it is sufficient only one of the automorphism group parameters to be time-dependent to reduce the metric at all times to the form (6.26); for the Bianchi types IV, VI_h and VII_h only two of the automorphism group parameters need to be time-dependent to reduce the metric at all times to the form (6.26). In the tilted case which is the general situation, in principle all the automorphism group parameters should be time-dependent; however in the simplest tilted case for the Bianchi models included in Table 3, only two of the automorphism group parameters need to be time-dependent to keep the metric in the form (6.26) at all times.

The Bianchi types II and V have a 6-dimensional automorphism group which an appropriate choice of their invariance families, allow further simplification of the metric given by (6.26a). The general vacuum and orthogonal type V cases must have one of the special automorphism group parameters constant, but there is a subsystem in which additionally another two special automorphism parameters are also time-independent. Type II vacuum and orthogonal cases have two time-independent special automorphism group parameters with a subsystem in which another special automorphism parameter is also time-independent. We notice that in type II the Ricci tensor component R_{03} vanishes at all times. This immediately says that in type II the fluid 4-velocity component $u_3 = 0$. Consequently type II Universes are vorticity-free although they may have a non-vanishing tilt angle (this result is in agreement with King and Ellis, 1973).

Although we present here the Einstein field equations for all non-abelian non-semisimple Bianchi Universes, it was our intention to stress the usefullness of the time-dependent automorphism group as a mechanism to simplify as much as possible the metric of these models, and present some qualitative analysis of the field equations rather than try to solve them in a hunt for new cosmological solutions. We believe that the application of this method is not yet exhausted and that further interesting solutions may become accessible through use of the techniques discussed in this Chapter.

A compact version of the material contained in Chapters 5 and 6 has recently been published in: **Axisymmetric Systems, Galaxies and Relativity, ed. M. A. H. MacCallum (Cambridge University Press, 1985).**

CHAPTER 7

ON THE NATURE OF THE INITIAL SINGULARITY

7.1 Introduction

One of the most intriguing problems in relativistic cosmology is the existence of space-time singularities. Roughly speaking a singularity is a place where the predictability of a theory breaks down. Physically a singularity occurs when some physical measurable quantities such as the energy-density or the temperature become infinite. not necessarily. In cosmology a singularity occurs when the world line of an observer $\lambda(s)$ cannot be extended beyond a finite distance s along the curve $\lambda(s)$. An observer that reaches the singularity simply cease to exist. A typical example of a cosmolocical singularity is the Big-Bang, however, many different types of space-time singularities are known.

Space-time singularities have been known for a long time as far as General Relativity is concerned, but only during the end of the sixties and beginning of the seventies was a great development achieved in the subject (see Tipler et. al. 1980 for an excelent review on singularities with historical remarks on the development of the theory). One of the major achievement has been the 1970 singularity Theorem of Hawking and Penrose which essentially states that (Hawking and Ellis, 1973):

If in a space-time (M, g) the following conditions are satisfied:

1) $R_{ab}K^aK^b > 0$, for all non-spacelike vectors K^a

2) There are no closed timelike curves

3) There exists a point from which the past null-cone starts to reconverge

4) All non-spacelike geodesics experience some curvature

then there exists an incomplete non-spacelike geodesic; in other words a space-time singularity.

The Hawking-Penrose Theorem predicts the existence of space-time singularities, however it does not give information about the nature of the singularity such as whether the singularity occurs in the past or in the future or how strong the singularity is. Thus most of the recent work on singularities has been concentrated on investigating their nature rather than their existence.

There are a variety of types of space-time singularities. Essentially they fall in one of the following categories (Ellis and Schmidt, 1977):

a) curvature singularities, which are characterized by the fact that some of the Riemann tensor components diverge when they are measured in a frame that is parallely propagated along a curve $\lambda(s)$.

b) non-curvature singularities, which occurs when all the Riemann tensor components are bounded in a parallely propagated frame along a curve $\lambda(s)$.

The curvature singularities are split into two classes which are called: i) scalar when some scalar polynomial field in the Riemann tensor diverges along $\lambda(s)$, and ii) non-scalar when no scalar polynomial field ~~in~~ s unbounded along the curve $\lambda(s)$. The Big-Bang is a scalar curvature singularity; many other examples of space-time singularities may be found in Ellis and King (1977). The tilted cosmological models exhibt the most interesting kind of singularities as can be seen in the papers of Siklos (1978) and Collins and Ellis (1979).

In this Chapter we are particularly interested in investigating some aspects of the nature of the initial singularity in the Friedmann-LeMaitre-Robertson-Walker (FLRW) Universe models.

Earlier you had used FRW; what is the difference between FRW & FLRW?

It is customary assumed that the initial singularity (Big-Bang) in the FLRW Universe models is an <u>isotropic</u> singularity (by isotropic singularity here we mean a singularity where the volume of a body is crusched isotropically to zero near the singularity) with strength the same as that of the singularity in the energy-density $\mu(t)$. However in this chapter we will show that not all particles in the universe experience an isotropic singularity. In order to examine this issue it is convenient to consider the time-reverse of particles emerging from the initial singularity in an expanding FLRW Universe. This corresponds to the case of particles falling into a singularity in the future, which is exactly what occurs in the final singularity in a closed FLRW Universe.

A FLRW Universe appear to be isotropic to any comoving observer with fundamental 4-velocity **u**. However in the hot early stages of the Universe near the singularity, almost all particles are moving at a very high speed relative to the fundamental velocity **u**. Consequently at any particular space-time event, the effective density of matter experienced by such particles is much higher than that experienced by a fundamental observer; and the effect of the space-time curvature on these particles is anisotropic.

Similar effects will occur in other more complex Universe models (such as Bianchi Universes). The singularity as experienced by a general particle is rather different than that experienced by a fundamental one because almost all particles are moving at high speed relative to the ~~average~~ velocity of motion in the Universe, which defines the velocity of a (notional) fundamental observer. ?
Thus both the strength of the singularity and its nature may differ from that which would be experienced by a fundamental observer. Our examination of FLRW Universes here gives an idea of the kinds of effects that may occur.

7.2 FLRW Geometry and Dynamics

The spatially-homogeneous and isotropic FLRW metric may be written as (Ellis, 1972)

$$ds^2 = - dt^2 + R^2(t)[dr^2 + f^2(r)(d\theta^2 + \sin^2\theta \, d\phi^2)], \tag{7.1}$$

where $R(t)$ is the 'Radius' (or scaling) function, $f(r) = (\sin r, \; r, \; \sinh r)$ according to the 3-spaces of curvature be (positive, zero, negative) respectively; and the fundamental velocity

$$u^a = \delta^a{}_o = (1, 0, 0, 0). \tag{7.2}$$

The energy-momentum tensor of the matter content is that of a perfect fluid (see section 2.4) with 4-velocity **u**,

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab}. \tag{7.3}$$

The energy density μ and the pressure p are given by

$$\mu = T_{ab}u^a u^b, \tag{7.4a}$$

$$p = (1/3)T_{ab}h^{ab}, \tag{7.4b}$$

where $h^{ab} = g^{ab} + u^a u^b$ is the projection tensor to the surfaces of homogeneity. These quantities will be assumed to satisfy the usual strong energy conditions,

$$\mu + p > 0, \tag{7.5a}$$

$$\mu + 3p > 0. \tag{7.5b}$$

In the hot early universe the matter is very well described by an ideal Fermi gas with equation of state

$$p = (1/3)\mu, \tag{7.6}$$

which corresponds to radiation (see section 2.4). Thus the particles will be moving fast on the average relative to u (their mean velocity will be u, but the root mean of the squares of their velocities relative to u will be high).

The conservation equation (eqs. 2.43) for the FLRW Universes reads

$$\dot{\mu} + 3(\mu + p)\dot{R}/R = 0. \tag{7.7}$$

From the conservation equation (7.7) and the equation of state (7.6) we have that

$$\mu = \mu_o(R_o/R)^4, \quad \mu_o = \text{const.} \neq 0. \tag{7.8a}$$

The temperature T is related to the energy-density by

$$\mu = aT^4, \tag{7.8b}$$

and so, from eq. (7.8a) we have

$$T = T_o/R, \quad T_o = \text{const.} \neq 0. \tag{7.8c}$$

The Einstein field equations determine the evolution of R(t). This is given by the solution of the Friedmann equation

$$3(\dot{R}/R)^2 - \mu + 3k/R^2 = 0, \tag{7.9}$$

where the constant k is equal to (+1, 0, -1) according to the 3-spaces being (closed, flat, open) respectively. At early times when R(t) is small and eq. (7.8a) holds, the asymptotic solution of eq. (7.9) is

$$R(t) \simeq t^{1/2}, \tag{7.10a}$$

showing that from (7.8a) and (7.8c)

$$\mu = \mu_0 (t_0/t)^2, \tag{7.10b}$$

$$T = T_0 (t_0/t)^{1/2}, \tag{7.10c}$$

where $\mu_0 = \mu(t_0)$ and $T_0 = T(t_0)$.

Fast moving observer. By virtue of the spatial homogeneity of the universe model, we can choose without loss of generality a galaxy $\mathcal{G}$ as the origin of the coordinates $r = 0$. Let us consider now a radial timelike geodesic $\mathcal{T}$ at the hyperbolic angle β to the fundamental velocity **u** with the tangent vector

$$U^a = \cosh\beta(t)\, u^a + \sinh\beta(t)\, e^a, \tag{7.11}$$

where e^a is a unit radial spacelike vector orthogonal to u^a and with components $e^a = (0,\ R^{-1},\ 0, 0)$. Suppose that at the intersection of the galaxy world line and the timelike geodesic $\mathcal{T}$ the hyperbolic angle β takes the value β_0, the time t the value t_0 and the radius function R the value $R_0 \equiv R(t_0)$. Then from eq. (7.1) the vector **U** is parallel propagated along itself, i.e. a geodesic vector field, iff (see eq. 2.18)

$$\nabla_U U = 0 \iff (R\sinh\beta)^{\cdot} = 0, \tag{7.12a}$$

which immediately says that

$$\sinh\beta(t) = R_0 \sinh\beta_0 / R(t). \tag{7.12b}$$

This is the tangent vector to the path that radially freely moving particles will take between collisions. The proper time τ measured along the geodesic is determined by the equation

$$dt/d\tau = U^0 = \cosh\beta, \tag{7.13a}$$

so we have

$$\tau = \int R(t)[R^2(t) + R_o^2\sinh^2\beta_o]^{-1/2}dt. \tag{7.13b}$$

Near the singularity (when R(t) is small) eq. (7.13b) gives

$$\tau = \int R(t)/R_o\sinh\beta_o\ dt, \tag{7.13c}$$

where by use of (7.10a) we have

$$\tau/\tau_o \simeq (t/t_o)^{3/2} \Rightarrow d\tau/dt \simeq (3/2)\,\tau_o\,(t/t_o)^{1/2}. \tag{7.13d}$$

So we see that $d\tau/dt \to 0$ as $t \to 0$. This implies the first effect experienced by a fast moving observer, namely: **The rate of change of any spatially homogeneous quantity measured by him will be slower than that measured by a fundamental observer** (because of their relative time dilatation). Thus in particular he will measure the radius function R, the energy density μ and the temperature T to change slower as he approachs the initial singularity than will a fundamental observer (for whom the proper time is t). Actually those ratio are given by

$$R/R_o \simeq (\tau/\tau_o)^{1/3} \Rightarrow (1/R_o)dR/d\tau \simeq (1/3)\,\tau_o^{-1/3}\tau^{-2/3}, \tag{7.14a}$$

$$\mu/\mu_o \simeq (\tau_o/\tau)^{4/3} \Rightarrow (1/\mu_o)d\mu/d\tau \simeq -(4/3)\,\tau_o^{4/3}/\tau^{7/3}, \tag{7.14b}$$

$$T/T_o \simeq (\tau_o/\tau)^{1/3} \Rightarrow (1/T_o)dT/d\tau \simeq -(1/3)\,\tau_o^{1/3}/\tau^{4/3}. \tag{7.14c}$$

The effective density. This first effect is compensated by a second effect. Let us introduce an orthonormal basis of vectors $\{E_a\}$ that is parallel propagated along the timelike geodesics and having $E_o = U$; for that is a 'physical' basis that would corresponds to measurements by an observer moving on the geodesics with tangent vector U. The basis $\{E_a\}$ is

$$E_o{}^a = U^a = (\cosh\beta,\ R^{-1}\sinh\beta,\ 0,\ 0), \tag{7.15a}$$

$$E_1{}^a = (\sinh\beta,\ R^{-1}\cosh\beta,\ 0,\ 0), \tag{7.15b}$$

$$E_2{}^a = (0,\ 0,\ (fR)^{-1},\ 0), \tag{7.15c}$$

$$E_3{}^a = (0,\ 0,\ 0,\ (fR\sin\theta)^{-1}), \tag{7.15d}$$

where $\beta(t)$ is given by eq. (7.12b). The effective energy density measured by the fast moving observer is

$$\mu' = T_{ab}E_o{}^aE_o{}^b = \mu\cosh^2\beta + p\sinh^2\beta; \tag{7.16a}$$

the effective pressure in the 1-direction is

$$p'_{11} = T_{ab}E_1{}^aE_1{}^b = p\cosh^2\beta + \mu\sinh^2\beta; \tag{7.16b}$$

and the effective energy flux is

$$q' = T_{ab}E_o{}^aE_1{}^b = (\mu + p)\sinh\beta\cosh\beta. \tag{7.16c}$$

Substituting from (7.12b) and (7.10a) give μ', p'_{11} and q' as functions of t and from (7.13) give as funnctions of τ. In particular, we find

$$\mu' = \mu + (\sinh\beta_o\ R_o/R)^2(\mu + p), \tag{7.17}$$

for the effective energy density μ'. Thus for small values of R(t) near the singularity we have

$$\mu'/\mu_o \simeq C_o\ (R_o/R)^6, \tag{7.18}$$

where $C_o = (4/3) \sinh^2\beta_o$. The expression (7.18) shows how at early times μ' is at each time much larger than μ (given by (7.8a)). If we substitute (7.10a) into (7.18) we get

$$\mu'/\mu'_o \simeq C'_o\ (t_o/t)^3, \tag{7.19a}$$

or in terms of τ

$$\mu'/\mu'_o \simeq C'_o\ (\tau_o/\tau)^2, \tag{7.19b}$$

where $\mu'_o = \mu_o(1 + 4/3 \sinh^2\beta_o)$ and $C'_o = (1 + 4/3 \sinh^2\beta_o)C_o$.

The interesting feature here is how **the two effects cancel out: μ'/μ'_o has the same dependence** in the proper time τ as μ/μ_o has in the proper time t. So both the fundamental observer and the fast moving observer measure the effective energy density to increase as the same power of their measured proper time. What happens is that at any particular event $t = t_o$ on τ, the fast moving observer measures a much higher effective energy density than the fundamental observer, but the time dilatation results in him measuring the same rate of increase of the effective energy density with his proper time. The fundamental observer attain the same level of density at an earlier cosmic time t' than the fast moving observer experiences at t; but both measure the same history for the energy density as a function of their proper time.

The null case. A particular situation occurs when the particles considered are taken to move at the speed of light; they will be moving along null geodesics. Let a future-directed radial null geodesic be $x^a(\lambda)$ with tangent vector

$$K^a = dx^a/d\lambda, \tag{7.20a}$$

then K^a satisfy

$$K^a K_a = 0 \text{ (null vector condition)}, \tag{7.20b}$$

$$K_{a;b}K^b = 0 \text{ (geodesic condition)}. \tag{7.20c}$$

The null vector K^a can be written from the metric (7.1) as

$$K^a = R_o/R\ (u^a + e^a) = (1 + z)(u^a + e^a), \tag{7.21a}$$

where z is the redshift measured at t_o for the radiation emitted at t.

In terms of the orthonormal basis $\{E_a\}$ given in (7.15), the null vector K turns out to be

$$K^a = R_o/R\ (\cosh\beta - \sinh\beta)(E_o{}^a + E_1{}^a). \tag{7.21b}$$

The affine parameter λ along the null geodesics can be found from (7.20a) according to

$$K^a = dx^a/d\lambda \quad \Rightarrow \quad K^o = dt/d\lambda,$$

but

$$K^o = -u_a K^a = R_o/R(t), \tag{7.22a}$$

thus we have

$$\lambda = \int R(t)/R_o\ dt = \int (1 + z)^{-1}\ dt. \tag{7.22b}$$

The effective energy density measured by the null particles is

$$\mu'' = T_{ab}K^aK^b = R_{ab}K^aK^b = (\mu + p)(R_o/R)^2. \tag{7.23a}$$

At early times μ'' takes the form

$$\mu'' = (4/3)\,\mu(R_o/R)^2 = (4/3)\,\mu_o\ (R_o/R)^6. \tag{7.23b}$$

This relation shows immediately that at early times the null particles and the fast moving particles measure essentially the same effective energy density. The equations (7.22b) and (7.10a) give

$$\lambda/\lambda_o = (t/t_o)^{3/2}, \tag{7.23c}$$

which agrees with the relation (7.13d). Similarly we have

$$\mu''/\mu''_o = (t_o/t)^3 = (\lambda_o/\lambda)^2, \tag{7.23d}$$

which also exibts the same power dependence measured along the null geodesics as eq. (7.19b) for the fast moving observer.

In some cases curvature measured along a parallel propagated frame is much higher than in some other frames. In fact it can be divergent in the parallel propagated frame and bounded in another (Ellis and Schmidt, 1977 and Collins and Ellis, 1979). In the present case the frame (7.15) is parallel propagated along the null geodesics parametrized by λ if we have

$$e^{\beta(t)} = e^{\beta_o}R(t)/R_o. \tag{7.24}$$

The energy density μ' evaluated in the parallel propagated null geodesic frame is

$$\mu' = T_{ab}E_o{}^aE_o{}^b = \mu\cosh^2\beta + p\sinh^2\beta, \tag{7.25}$$

where now the angle $\beta(t)$ is given by equation (7.24) instead of by (7.12b). For small values of R(t) near the singularity we have from (7.25) that

$$\mu' \simeq \mu R_o^2 e^{-2\beta_o}/R^2, \tag{7.26a}$$

which from (7.8a) gives

$$\mu' \simeq C_o (R_o/R)^6, \tag{7.26b}$$

where $C_o = (1/3) \mu_o e^{-2\beta_o}$. Thus the effective energy density in the parallel propagated frame along the null geodesics presents the same effect here as that discussed earlier on (7.18). Of course, (7.26b) expressed in terms of the affine parameter λ along the null geodesics also shows the same result as eq. (7.19b). So we can see that in a parallel propagated frame along the null geodesics $\mu'/\mu'_o \sim (t_o/t)^3$, but there is another frame along the timelike geodesics (namely, that one in which $\beta(t) = 0$) for which $\mu'/\mu_o \sim (t_o/t)^2$ holds. In fact it is clear from eq. (7.16a) that the latter frame is the <u>minimal</u> frame for such measurements. Any other frame will give larger answer for μ'. Thus the singularity in the Ricci tensor (scalar singularity) is considerably stronger when measured in an orthonormal frame parallel propagated along the null geodesics, than in the minimal frame (which is related to the parallel propagated frame by a Lorentz transformation that diverges when one approaches the initial singularity). Nevertheless, the effect is not as drastic as in the case of whimper singularities (Ellis and King, 1974), where the energy density is finite in the minimal frame but diverges in the parallel propagated frame. In the present case, the energy density in the parallel propagated frame is simply more rapidly divergent than in the minimal frame.

The relation between the two points discussed here is that when the energy density is evaluated relative to the null geodesic vector K^a, one is effectively using a parallel propagated frame (because K^a is parallel propagated). Thus the strength of the effective Ricci tensor singularity agrees in these two cases.

7.3 The Anisotropic Effect of Curvature

In the previous section we have established the existence of an 'equal strength' singularities in the Ricci tensor for both the fundamental observer and the fast moving observer when measured in their proper time. The aim of this section is to consider what effect these curvature singularities have on the particles considered.

The tidal effects. We are exploring here this issue in an entirely classical context. So the curvature tensor acquires its main physical meaning as the source term in the geodesic desviation equation (Pirani, 1956) and hence the driving term causing gravitational field (tidal forces) stresses in elastic bodies (Weber, 1961, and Misner, Thorne and Wheeler, 1973). Thus in the present case it can be regarded as causing tidal force stresses to the particles falling into the singularity. Using an elastic approximation for the responce of this body, the desviation vector ξ expressing the strain in the body in the direction r satisfies the equation (Weber, 1961)

$$m d^2\xi^f/dt^2 + D^f{}_a d\xi^a/dt + H^f{}_a \xi^a = - m R^f{}_{abc} V^a \xi^b V^c, \tag{7.27}$$

where V^a is the 4-velocity of the body; the damping force is given by the second term and the restoring force by the third term with $D^f{}_a$ and $H^f{}_a$ tensors describing the elastic properties of the body. Thus the driving term for the strain is the Riemann tensor term

$$\Sigma^f{}_b = R^f{}_{abc} V^a V^c.$$

This is the term that we will investigate. Let us consider the Einstein field equations in the form

$$R_{ab} = \kappa(T_{ab} - (1/2)\,Tg_{ab}) + \Lambda g_{ab}, \tag{7.28}$$

where κ is the graviational constant and Λ is the cosmological constant.

The Weyl conformal tensor is defined by (Ellis, 1973)

$$C_{abcd} = R_{abcd} + (1/2)(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}) + {}$$
$$+ (R/6)(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{7.29a}$$

should be minus (-)

If we subistitute from (7.28) in (7.29a) we have

$$R_{abcd} = C_{abcd} + (\kappa/2)(T_{ac}g_{bd} + T_{bd}g_{ac} - T_{ad}g_{bc} - T_{bc}g_{ad}) - {}$$
$$- (1/3)(\kappa T^a{}_a - \Lambda)(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{7.29b}$$

this equation is correct

which is in fact completly equivalent to the Einstein field equations (7.28). If we assume a perfect fluid matter source, equation (7.29b) turns out to be

$$R_{abcd} = C_{abcd} + (\kappa/2)(\mu + p)(u_a u_c g_{bd} + u_b u_d g_{ac} - u_a u_d g_{bc} - {}$$
$$- u_b u_c g_{ad}) + (1/3)(\kappa\mu + \Lambda)(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{7.29c}$$

Contracting equation (7.29c) twice with the vector V^a, where $V_a V^a = -1$ and $V^a u_a = -\cosh\beta$, we have

$$\Sigma_{ac} = R_{abcd}V^b V^d = E_{ac} + \kappa/2\ (\mu + p)[-u_a u_c + \cosh\beta(u_a V_c + u_c V_a) + {}$$
$$+ \cosh^2\beta\ g_{ac}] - 1/3\ (\kappa\mu + \Lambda)(g_{ac} - V_a V_c), \tag{7.30}$$

where $E_{ac} = C_{abcd}V^b V^d$ is the `eletric part' of the Weyl tensor (see Ellis, 1973).

Equation (7.30) gives the form of the driving term Σ_{ac} whenever the field equations are satisfied with a perfect fluid matter content. In particular, in a spacelike direction r ($r^a r_a = 1$) that is perpendicular to the fluid flow vector u in the rest frame of V, we have

$$\Sigma_{ac} r^a r^c = E_{ac} r^a r^c + 1/3\,[\kappa/2\,(\mu + 3p) - \Lambda] + \kappa/2\,(\mu + p)\sinh^2\beta, \qquad (7.31a)$$

while in the spacelike direction r parallel to the projection of u in the rest frame of V we have

$$\Sigma_{ac} r^a r^c = E_{ac} r^a r^c + 1/3\,[\kappa/2\,(\mu + 3p) - \Lambda]. \qquad (7.31b)$$

The direction-avareging of $\Sigma_{ac} r^a r^c$ over all directions orthogonal to V^a is given by

$$\langle \Sigma_{ac} r^a r^c \rangle = 1/3\,[\kappa/2\,(\mu + 3p)] + \kappa/3\,(\mu + p)\sinh^2\beta, \qquad (7.31c)$$

which is the source term in the Raychaudhuri equation for matter moving with 4-velocity V^a, determining the volume evolution of such matter (Ehlers, 1971 and Ellis, 1972).

FLRW Universes. Let us now see what happens to a particle as it falls at a very high speed into a singularity in a FLRW Universe. In this case, as FLRW Universes are conformaly flat, we have

$$C_{abcd} = 0 \;\Rightarrow\; E_{ac} = 0.$$

At any particular space-time position near the singularity, the particles experiences a tidal force given by eq. (7.30), where $V_a = U_a$ is given by (7.11) and the angle β is given by (7.12b). Thus from eqs. (7.31b), (7.10b) and (7.14b) the fast moving observer will experience near the singularity a tidal force compressing him along the direction of motion with magnitude of

$$\Sigma_{ab}r^a r^b = \kappa\mu_o/3\ (t_o/t)^2 = \kappa\mu_o/3\ (\tau_o/\tau)^{4/3}. \tag{7.32a}$$

On the other hand, in the two orthogonal direction he experiences a tidal force given by (7.31a) which will be much larger (because of the $\sinh^2\beta$ factor, which diverges at the singularity). For this case, near the singularity we have

$$\Sigma_{ab}r^a r^b = 2\kappa\mu_o/3\ \sinh^2\beta_o\ (t_o/t)^3 = 2\kappa\mu_o/3\ \sinh^2\beta_o\ (\tau_o/\tau)^2. \tag{7.32b}$$

The magnitude of the anisotropy of $\Sigma_{ab}r^a r^b$ when r^a ranges over all directions orthogonal to u_a is

$$\Omega = \kappa/2\ (\mu + p)\sinh^2\beta. \tag{7.33}$$

The average volume behavior is determined by the directional average (7.31c), exactly as would have been expected from the Raychaudhuri equation for particles moving with 4-velocity V_a. Near the singularity, this is given approximately by $2/3\ \Sigma_{ab}r^a r^b$ which by (7.32b) is (up to a constant) the same expression given by (7.19b) for μ'. This expression confirms what we have found in the previous section: while the energy density is much larger at any space-time event for the fast moving observer than the stationary ones, it is the same function of proper time for both particles. The major difference comes from the inequality of the forces along and perpendicular to the direction of relative motion for the fast moving particles, which will cause severe distortion effects to the particle, compressing it perpendicular to the direction of motion in a much larger magnitude than sideways, while its volume is crushed to zero. For the fundamental observer its volume is crushed to zero as well but without distortion. According to Tipler (1977) this characterizes a `strong singularity'.

The null case. If we contract now eq. (7.29a) twice with the null vector K^a, one obtains

$$\Sigma''_{ab} = E''_{ab} + \kappa/2\ (\mu + p)[(u^cK_c)^2 g_{ab} - (u^cK_c)(u_aK_b + u_bK_a)] -$$

$$- \ 1/3\ (\kappa\mu + \Lambda)K_aK_b, \qquad (7.34)$$

where $\Sigma''_{ac} = R_{abcd}K^bK^d$ and $E''_{ac} = C_{abcd}K^bK^d$. Then as required from the Riemann tensor symmetries, $\Sigma''_{ac}K^c = 0$, hence the driving term for parallel distortion is

$$\Sigma''_{ab}K^aK^b = 0,$$

which is consistent with the fact that null radiation is transverse. A distortion-free focussing takes place in the orthogonal 2-spaces (spanned by the vector r^a, where $r^ar_a = 1$, $r^aK_a = 0$, $r^au_a = 0$), with magnitude determined by

$$\Sigma''_{ab}r^ar^b \quad = E''_{ab}r^ar^b \quad + \kappa/2\ (\mu + p)(u_aK^a)^2. \qquad (7.35a)$$

Hence in the case of FLRW Universe near the singularity (when $E''_{ab} = 0$)

$$\Sigma''_{ab}r^ar^b \quad = 2\kappa\mu_o/2\ (R_o/R)^6 = 2\kappa\mu_o/3\ (\lambda_o/\lambda)^2. \qquad (7.35b)$$

This implies that the effect of the gravitational field in focussing light rays is (up to a constant) the same as its converging effect on the fast moving particles, except that it is an isotropic focussing perpendicular to the direction of motion without distortion (there is no compression along the direction of motion in the case of zero-rest mass particles). As in the case of massive particles, the effective strength of the curvature is that measured in a parallel propagated frame, not in the fluid rest frame.

7.4 Concluding Remarks

In this chapter we concentrated attention on the nature of the initial singularity in the FLRW Universes. We have shown that the effect of the FLRW singularity (Big-Bang) on the particles in it is significantly affected by the fact that near the singularity, temperatures are very high and on the average the particles are moving at a very high speed in respect to the surfaces of homogeneity. This means that considering a particle falling into the singularity in the time-reverse sense (as in the final singularity for the closed FLRW Universe) at any particular time t_o in the standard FLRW coordinates, the fast moving particles experiences much higher gravitational tidal forces acting to compress them than a fundamental particle (which is at rest relative to the surfaces of homogeneity). Nevertheless, the time dilatation between these two observers acts to counteract this effect: the effective energy density experienced by the fast moving observer near the singularity is the same function of his proper time as that measured by a fundamental observer is of his proper time. The major difference experienced between them is that the fast moving particles are <u>anisotropically</u> compressed; they are compressed orthogonal to their direction of motion by much greater tidal gravitational forces than those compressing them in the direction parallel to the motion. If we allow the slowing down of the fast moving particles due to collisions, the conclusion will be unaltered: the effective average speed of motion of the particles will be reduced, but between collisions the situation will be exactly as just described. Consequently the overall effect is unchanged.

The implication for the particles depend on the model considered. Classically, they would respond according to eq. (7.27), being infinitly distorted as they are crushed to zero volume by anisotropic tidal forces. If one allows for the change of direction with collisions, the effective direction of distortion axes will change at each collision, leading to successive anisotropic compression taking place in randomly distributed directions in the particles. The implications for particles modeled in a quantum theoretic way should be similar: they are subject to divergent anisotropic forces rather than isotropic ones. Similarly, a group of particles moving at very high speed in the same direction will be compressed together in the same manner.

In practice, at the initial singularity this effect will take place in the foward direction of time. The tidal gravitational forces will decrease as time progress in the reverse direction. They will be indefinitly large arbitrarily near the singularity, and will be much larger at any particular time (and so at any particular energy density μ or temperature T) for the fast moving observer, but decreasing in the same way with proper time for both the fundamental and fast moving observers. Similarly, the initial singularity in Bianchi cosmologies (e.g. the cigar and pancake singularities in Bianchi I models (Thorne, 1967) and in power law singularities in more general cases (Wainwright, 1984)) will also look different from the view point of a fast moving particles than from the usual non-tilted reference frame.

The effects of curvature at the final singularity in the closed (k = +1) FLRW Universe will be identical to these described above. The tidal gravitational forces will be increased in the forward direction of time (just as for an observer falling into a black hole, cf. Misner, Thorne and Wheeler, 1973).

The physical implications are not clear without detailed models for the particles, but the effect should be remembered in considering physical process near the initial or the final singularity. These effects may not necessarily have serious implications for the internal structure of the particles or their interactions with each other, but certainly it changes one's view of the effective action of the singularity from the usual picture of an isotropic compression.

The material of this chapter has been recently published in the General Relativity and Gravitation Journal, vol. 17, 397, (1985). In the published version two corrections should be done: firsly in section 2, item 2.2, for the first effect found by a fast moving observer we should replace 'faster' by 'slower' and secondly the null vector K^a given in item 2.4 should be read as in equation (7.21b) of this chapter.

CHAPTER 8

THE LUMPY UNIVERSE

8.1 Introduction

It is a normal procedure in cosmology to consider the Universe as being spatially homogeneous on a large scale (this is the assumption underlying the spatially homogeneous cosmologies), and often the additional condition of spatial isotropy is assumed, leading to the FLRW Universe models. Both conditions are indeed well satisfied when the Universe is observed over scales greater than 100 Mpc ($\sim 10^8$ light years, Misner et al. 1973). However for scales of less than 100 Mpc. the Universe is far from being spatially homogenous and isotropic. In fact, it appears quite inhomogeneous on a smaller scale where the stars and galaxies are observed as concentrations of matter between vast empty regions of the Universe.

For many purposes in cosmology, these local inhomogeneities are irrelevant. Thus it is customary to ignore these small scale structures while investigating the geometry and dynamics of the Universe as a whole. The simplest and most convenient approach is to consider a homogeneous distribution of matter where the local inhomogeneities have been "smoothed-out" and this homogeneous distribution described as a perfect fluid source for the Einstein field equations. The FLRW Universe models are our 'best-fit' Universe models under these considerations (there are some other good candidates such as the Bianchi type I and V, also Type IX, although they ~~are~~ may be anisotropic Universe models).

Some more realistic Universe models have been considered where the spatial homogeneity condition has been dropped. These inhomogeneous models are rather complicated, as they involve spatial dependence in addition to the time dependence. Many of them have been shown to start off as inhomogeneous and evolve towards homogeneous models (see for instance Bonnor and Tomimura, 1976). However, the transition from being inhomogeneous to homogeneous is not due to a smoothing-out procedure but a result of the dynamical evolution of the model ~~in itself from~~ early to late times. There may be some sort of physical process which causes this transition, such as viscosity, but these processes do not deal with the problem of smoothing-out inhomogeneities on a spatial hypersurface at a given time t. To examine this issue it would be interesting to consider a cosmological model in which the spacelike sections are inhomogeneous on a small scale but become homogeneous on a large scale. The smoothing-out of this local lumpiness could then be responsible for a transition to smoothness with time.

So far in cosmology the smoothing-out problem has been little considered. Very recently, George Ellis at the 10th. General Relativity and Gravitation International conference has called some attention to the problem (see Ellis' article in the Proceedings of the GRG 10 conference ed. by B. Bertotti et al. 1984). His idea is to consider different scales of description of the same physical system. Suppose that at a scale of description where the stars are individually perceived, the metric and the ~~energy momentum~~ stress energy tensors are given by (g_{1ij}, T_{1ij}), where we call this scale 1. This level of description involves 10^{22} stars; it is far too detailed for cosmological purposes. Thus a coarser scale of description can be considered where the metric and ~~energy momentum~~ stress-energy tensors at this scale (g_{3ij}, T_{3ij}) are the respective smoothed-out quantities of scale 1. The scale 3 can represent galaxies and clusters of galaxies. It is still very detailed for many cosmological purposes, thus a scale 5 of description can be considered where all the galaxies and clusters of galaxies have been smoothed-out, providing a homogeneous picture of the Universe. The metric and the energy momentum at scale 5 are given (g_{5ij}, T_{5ij}), where these are the smoothed-out quantities of scale 3. At this scale of description the cosmological model can suitably be identified with the spatially homogeneous cosmological models.

Why not scales 2, 4, etc.

It has been suggested (Ellis, 1984) that there might be smoothing-out operators S_{ab} and S'_{ab} which carry the metric and the energy momentum tensors from one scale to another, such that

$$S_{31}(g_{1ij}) = g_{3ij} \; ; \quad S'_{31}(T_{1ij}) = T_{3ij},$$

why should you not put $S_{31}(g_{1ij}, T_{1ij}) = (g_{3ij}, T_{3ij})$?

$$S_{53}(g_{3ij}) = g_{5ij} \; ; \quad S'_{53}(T_{3ij}) = T_{5ij},$$

where the smoothing-out operators S_{ab} and S'_{ab} would satisfy some properties like

$$S_{aa}(g_{aij}) = g_{aij} \; ; \quad S'_{aa}(T_{aij}) = T_{aij},$$

$$(S_{ab} \circ S_{bc})(g_{cij}) = S_{ac}(g_{cij}),$$

$$(S'_{ab} \circ S'_{bc})(T_{cij}) = S'_{ac}(T_{cij}), \text{ (a, b, c labels the scales).}$$

If the Einstein field equations hold for the scale 1 (this is a reasonable statement because most of the experimental test of General Realtivity are based on this scale), the ~~metric~~ metric g_{1ij} will be determined by

$$G_{1ij} = R_{1ij} - (1/2)R_1 g_{1ij} = T_{1ij},$$

where R_{1ij} are the Ricci tensor components and R_1 is the Ricci scalar for the metric g_{1ij}.

The application of the smoothing-out operators S_{31} and S'_{31} to g_{1ij} and T_{1ij}

respectively will determine the metric and energy momentun of scale 3. However, in principle the Einstein field equations

$$G_{3ij} = T_{3ij},$$

~~will~~ may not be satisfied because the Einstein tensor G_{3ij} obtained by smoothing-out G_{1ij} in general would not correspond to the Einstein tensor G_{3ij} obtained from the smoothing-out metric g_{3ij}. Similarly, this problem also occurs from scale 3 to scale 5. Thus a correction (or compensating) term P_{3ij} is defined representing the smoothed-out difference between the Einstein tensor at scale 1, G_{1ij}, and the Einstein tensor G_{3ij} obtained from the smothed-out metric g_{3ij}. Then, at scale 3 and 5 the field equations become

$$G_{3ij} = T_{3ij} + P_{3ij},$$

$$G_{5ij} = T_{5ij} + P_{5ij}.$$

The contribution of the tensor P_{ij} corresponds to effects of the small scale inhomogeneities of the Universe on the dynamics of the smoothed-out scales. It is normally written in the right hand side of the field equations because it represents an effective source contribution to the smoothed-out scale (it can be thought of as a back reaction of the geometry).

In this chapter we will consider a lumpy Universe with two distinct observers: a small scale (SS) observer, who can only measure the small scale properties of the mediun, and a large scale (LS) observer, who can only see the large scale properties of the medium. We adapt the work of Noonan (1984) to determine the relations between the field equations for the SS medium to the LS medium. The correction tensor P_{ij} that should be introduced in the field equations of the LS medium and the smoothed-out energy momentum tensor seen by the LS observer are presented, and some corrections to Noonan's work are made.

8.2 The Smoothing-out Operator

Let us here give our attention to the smoothing-out problem between, say scale 3 and scale 5. From now on scale 3 will be refered as the small scale (SS) and its quantities denoted with a prime (e.g. μ', g'_{ij}, etc.), and scale 5 will be refered as the large scale (LS) or smoothed-out scale, with quantities denoted by (μ, g_{ij}, etc.).

The two-scale approach is a very common procedure in various branchs of physics. Typical examples of such approach are found in Thermodynamics and Statistical Mechanics. The simplest way to tackle the problem is to consider the large scale observer as someone who can only see an "average" of the small scale effects. According to this principle, let us define a smoothing-out operator S an an averaging operator over the small scale quantities (see Noonan, 1984):

$$S(Q') = \langle Q' \rangle = (1/V)\int Q'(g)^{1/2}d^4x, \tag{8.1}$$

why not SS?

where $V = \int (g)^{1/2}d^4x$ and g is the determinant of the LS metric. We assume that the 4-dimensional region of integration is large compared to a typical SS length scale and small when compared to a typical LS length scale. The term $(g)^{1/2}$ has been included in the definition (8.1) to make S coordinate invariant. However, as pointed out by Noonan (1984), it does not vary significantly over the region of integration. Thus we may consider S as being

$$S(Q') = \int Q'd^4x \,/ \int d^4x. \tag{8.2}$$

The smoothing-out operator ($S = \langle\ \rangle$) commutes with the covariant derivatives not obvious – explain

with respect to the LS metric (see Noonan, 1984). In other words,

$$S(Q'_{;i}) = \langle Q'_{;i} \rangle = \langle Q' \rangle_{;i} = [S(Q')]_{;i}. \tag{8.3}$$

Of course, it is assumed that the operator S just acts on the SS quantities and its action on the LS quantities is such that $S(Q) = Q$.

Let us consider the line element ds^2 corresponding to a coordinate displacement dx^i which is small on the SS level:

$$ds^2 = g'_{ij}dx^i dx^j. \tag{8.4}$$

If we take dx^i as a constant displacement, the averaging of equation (8.4) by use of eq. (8.2) gives

$$\langle ds^2 \rangle = \langle g'_{ij} \rangle dx^i dx^j. \tag{8.5}$$

The displacement dx^i is small for the SS observer and far too small to be perceived by the LS observer. If we consider now a displacement Δx^i which is small for the LS observer and large for the SS observer, the line element Δs^2 for the LS observer should have the same form as in eq. (8.5)

$$\Delta s^2 = \langle g_{ij} \rangle \Delta x^i \Delta x^j. \tag{8.6}$$

However, if the LS metric is g_{ij}, thus the LS observer would identify the line element Δs^2 as

$$\Delta s^2 = g_{ij}\Delta x^i \Delta x^j. \tag{8.7}$$

Hence, if we compare these two expressions for Δs^2 we conclude that

$$g_{ij} = \langle g'_{ij} \rangle. \tag{8.8}$$

This relation shows that the LS metric g_{ij} is obtained by smoothing-out the SS metric g'_{ij}, that is $g_{ij} = S(g'_{ij})$.

The difference between the SS metric g'_{ij} and the LS metric g_{ij} defines a new quantity

$$h_{ij} = g'_{ij} - g_{ij}, \tag{8.9}$$

which is not affected by the smoothing-out operator S. In fact,

$$S(h_{ij}) = \langle h_{ij} \rangle = \langle g'_{ij} - g_{ij} \rangle,$$

but from eqs. (8.2) and (8.8) we have that

$$S(h_{ij}) = \langle h_{ij} \rangle = 0. \tag{8.10}$$

Therefore, the difference between the SS metric and LS metric may vary considerably over a certain region of the space-time, nevertheless the average effect on the whole space-time vanishes.

The SS observer and the LS observer have different perceptions concerning to what they measure. As an example, the matter density measure by the SS observer differs from that measured by the LS observer. Suppose that μ' is the matter density measured by the SS observer in a certain region of the space-time; it is defined by

$$\mu' = dm'/dv', \tag{8.11}$$

where dm' is the SS element of matter enclosed by the proper spatial volume element dv'. The 4-volume of the SS observer is given by

$$d\tau' = ds'\ dv', \tag{8.12}$$

where ds' is the SS proper time. Since $d\tau'$ is a scalar, it is also given in a general coordinate form by

$$d\tau' = (g')^{1/2}\, d^4x. \tag{8.13}$$

The LS observer sees for the same space-time region a 4-volume which is given by

$$d\tau = ds\, dv, \tag{8.14}$$

where ds is the LS proper time and dv its spatial volume element. Similarly to the SS observer, the 4-volume of the LS observer $d\tau$ is also given by

$$d\tau = (g)^{1/2}\, d^4x. \tag{8.15}$$

If we now divide eq. (8.12) by (8.14) and make use of eqs. (8.11), (8.13) and (8.15) we finally get

$$dm'/dv = \mu'\, (g'/g)^{1/2}\, ds/ds', \tag{8.16}$$

but for the SS and LS observers, their proper time are given by

$$ds' = dt/u'^{o}, \tag{8.17a}$$

$$ds = dt/u^{o}. \tag{8.17b}$$

Thus, if we substitute these relations into eqs. (8.16) we have that

$$dm'/dv = \alpha\, \mu', \tag{8.18a}$$

where α is defined as

$$\alpha = (g'/g)^{1/2}\, u'^{o}/u^{o}. \tag{8.18b}$$

The LS observer measures then a proper density $\tilde{\mu} = \alpha\mu'$ for the same mass that the SS observer measures as having proper density μ'. It is worthwhile mentioning that the proper density $\tilde{\mu}$ measured by the LS observer is different from μ' because the two observers may have different motions; and also $\tilde{\mu} \neq \mu$ (=

dm/dv) because $\tilde{\mu}$ is based on what the SS observer regards as matter, while μ is based on what the LS regards as matter.

The Eisntein field equations for the SS and LS observers are, respectively

$$G'_{ij} = T'_{ij}, \tag{8.19a}$$

$$G_{ij} = T_{ij}. \tag{8.19b}$$

Thus the difference of these two equations gives

$$T_{ij} = T'_{ij} - \delta G_{ij}, \tag{8.20}$$

where $\delta G_{ij} = G'_{ij} - G_{ij}$. The quantity on the left hand side of equation (8.20) is a LS quantity which is not affected by the smoothing-out operator S defined in eq. (8.2). However the SS quantities on the right hand side are affected by the smoothing-out operator S. Thus, smoothing-out eq. (8.20) we have

$$T_{ij} = \langle T'_{ij} \rangle - P_{ij}, \tag{8.21}$$

where $P_{ij} = \delta G_{ij}$. It is clear from eq.(8.21) that the correct field equations for the LS medium is given with the matter source being the smoothed-out SS energy momentum contribution plus the correction term contribution.

The simplest way to find the effects proceeding from the difference P_{ij} is to consider an expansion of this contribution in terms of the difference h_{ij} defined in eq. (8.9). This will be carried out in the following section.

8.3 The LS Energy Momentum

We shall expand the right hand side of equation (8.21) in terms of the difference h_{ij}. The expansion will be taken up to second order contributions, assuming that higher order effects are too small to be perceived by the LS observer. The small scale metric can be expressed as

$$g'_{ij} = g_{ij} + h_{ij}, \tag{8.22a}$$

and its inverse

$$g'^{ij} = g^{ij} - h^{ij}. \tag{8.22b}$$

The raising and lowering of indices and the covariant derivatives are calculated with respect to the LS metric g_{ij}. The expansion of the symmetric connection coefficients (Christoffel symbols) given in eq. (2.29) gives

$$\Gamma'^{i}{}_{jk} = \Gamma^{i}{}_{jk} + Q^{i}{}_{jk} - S^{i}{}_{jk} + O^{3}(h), \tag{8.23a}$$

where we have defined the quantities

$$Q^{i}{}_{jk} = 1/2\ g^{il}[h_{lk;j} + h_{jl;k} - h_{jk;l}], \tag{8.23b}$$

$$S^{i}{}_{jk} = 1/2\ h^{il}[h_{lk,j} + h_{jl,k} - h_{jk,l}]. \tag{8.23c}$$

The Ricci tensor components (eq. 2.37) become

$$R'_{ij} = R_{ij} - g^{kl}H_{ijkl} + h^{kl}H_{ijkl} -$$

$$- g^{kl}g_{mn}(Q^{m}{}_{ij}Q^{n}{}_{kl} - Q^{m}{}_{ik}Q^{n}{}_{jl}) + O^{3}(h), \tag{8.24a}$$

It is NOT consistent to keep only first order terms in (8.22b) but compute to 2nd order in (8.23a) and elsewhere.

where we have defined

$$H_{ijkl} = 1/2\ [h_{ij;kl} + h_{kl;ij} - h_{ik;jl} - h_{jk;il}]. \tag{8.24b}$$

Thus, the difference of the SS Ricci tensor and the LS Ricci tensor components can be written as

$$\delta R_{ij} = R'_{ij} - R_{ij} = r_{ij} + q_{ij} + O^3(h), \tag{8.25a}$$

with

$$r_{ij} = - g^{kl} H_{ijkl}, \tag{8.25b}$$

$$q_{ij} = h^{kl} H_{ijkl} - g^{kl} g_{mn} [Q^m{}_{ij} Q^n{}_{kl} - Q^m{}_{ik} Q^n{}_{jl}]. \tag{8.25c}$$

The terms r_{ij} and q_{ij} represent the first and second order expansion contribution for the difference δR_{ij}, respectively. The difference δG^{ij} is given by

$$\delta G^{ij} = g'^{ik} g'^{jl} G'_{kl} - g^{ik} g^{jl} G_{kl},$$

these terms should be kept.

which using the eqs. (8.22) and (8.25a) becomes

$$\delta G^{ij} = g^{ik} g^{jl} \delta R_{kl} - 1/2\, g^{ij} g^{kl} \delta R_{kl} + 1/2\, h^{ij} g^{kl} \delta R_{kl} -$$
$$- 1/2\, g^{ij} h^{kl} \delta R_{kl} - g^{ik} h^{jl} \delta R_{kl} - g^{jk} h^{il} \delta R_{kl} + O^3(h), \tag{8.26}$$

where the contributions of the form $h^{ik} h^{jl} R_{kl}$ have been omitted as they happen in a much finer scale than those given by the third to the sixth term in eq. (8.26). The linear terms in the difference have also been omitted as they give no smoothed-out conribution. The correction term P_{ij} defined in eq. (8.21) is obtained when the expression for the difference δG^{ij} is smoothed-out. The terms involving linear contributions of the form $\langle h_{ij;k} \rangle$ and $\langle h_{ij;kl} \rangle$ vanish as a consequence of eq. (8.3). Thus using these results plus eq. (8.25a) we have

Ok

$$p^{ij} = \langle \delta G^{ij} \rangle = \langle q^{ij} - 1/2\, g^{ij}q^{k}{}_{k} + r_{kl}(1/2\, g^{ij}h^{kl} + 1/2\, g^{kl}h^{ij} -$$

$$- g^{ik}h^{jl} - g^{jk}h^{il}) \rangle + O^3(h). \tag{8.27}$$

This represents the expansion of the second term of eq. (821) in terms of the difference h_{ij}.

The first term in eq. (8.21) corresponds to the smoothed-out contribution of the SS energy momentum to the LS ~~energy momentum~~ stress energy tensor. This contribution should also be exoanded in terms of the difference h_{ij}. To do that, let us define the synchronous SS 4-velocity

$$V'^{i} = u'^{i}/u'^{o}, \tag{8.28}$$

where u'^{i} is the SS fluid 4-velocity and $u'^{i}u'_{i} = -1$ and $V'^{i}V'_{i} = (u'^{o})^{-2}$. From the relation $g'_{ij}u'^{i}u'^{j} = -1$ and using eq. (8.28) we get that

$$(u'^{o})^{-2} = \gamma^{-2} - h_{ij}V'^{i}V'^{j}, \tag{8.29a}$$

where

$$\gamma^{-2} = - g_{ij}V'^{i}V'^{j}. \tag{8.29b}$$

Expanding eq. (8.29a) up to second order in the difference h_{ij} results

$$u'^{o} = \gamma[\, 1 + 1/2\, \gamma^2 h_{ij}V'^{i}V'^{j}(1 + 3/4\, \gamma^2 h_{kl}V'^{k}V'^{l})] + O^3(h). \tag{8.30}$$

In a similar manner we may define the LS synchronous 4-velocity as

$$V^{i} = u^{i}/u^{o}, \tag{8.31}$$

which leads to

$$(u^{o})^{-2} = g_{ij}V^{i}V^{j}. \tag{8.32}$$

The ratio between the determinant of the SS metric and LS metric is given in terms of h_{ij} by

$$g'/g = 1 + h^{i}{}_{i} + 1/2\,[(h^{k}{}_{k})^{2} - h^{ij}h_{ij}] + O^{3}(h). \tag{8.33}$$

The ~~energy momentum~~ stress energy tensor for the SS medium is written as

$$T'^{ij} = \mu' u'^{i} u'^{j} + N'^{ij}, \tag{8.34}$$

where μ' is the SS energy density and N'^{ij} corresponds to the mechanical stress of the SS medium and peharps other fields in it. Thus the first term in the right hand side of eq. (8.34) is given by

$$\mu' u'^{i} u'^{j} = \tilde{\mu}\,\alpha^{-1}(u'^{o})^{2} V'^{i} V'^{j}, \tag{8.35a}$$

where we have used eq. (8.28) and the definition $\tilde{\mu} = \alpha\mu'$. If we now use eqs. (8.18b), (8.30) and (8.35a) we get

$$\begin{aligned} \mu' u'^{i} u'^{j} = \mu\gamma u^{o} V'^{i} V'^{j}[\, 1 -1/2\,(h^{k}{}_{k} - \gamma^{2} h_{kl} V'^{k} V'^{l}) + \\ + 1/4\, h^{kl} h_{kl} + 1/4\, h^{k}{}_{k}(1/2\, h^{l}{}_{l} - \gamma^{2} h_{mn} V'^{m} V'^{n}) + \\ + 3/8\,\gamma^{4}(h_{kl} V'^{k} V'^{l})^{2}] + O^{3}(h). \end{aligned} \tag{8.35b}$$

The LS energy momentum tensor is obtained by combining eqs. (8.21), (8.27), (8.34) and (8.35b), which tuns out to be

$$T^{ij} = \langle\, \tilde{\mu}\gamma u^{o} V'^{i} V'^{j} \,\rangle + \langle\, N'^{ij} \,\rangle + T_{G}{}^{ij} + O^{3}(h), \tag{8.36a}$$

where

$$T_G{}^{ij} = - P^{ij} - 1/2 \langle \tilde{\mu}\gamma u^o v'^i v'^j (h^k{}_k - \gamma^2 h_{kl} v'^k v'^l) \rangle +$$

$$+ \langle \tilde{\mu}\gamma u^o v'^i v'^j [1/2\, h^k{}_k (h^l{}_l - 2\gamma^2 h_{mn} v'^m v'^n) +$$

$$+ h^{kl} h_{kl} + 3/4\, \gamma^4 (h_{kl} v'^k v'^l)^2] \rangle. \tag{8.36b}$$

The LS energy momentum tensor given by eq. (8.36a) can be interpreted as the sum of three different contributions: i) the first term corresponds to the SS smoothed-out contribution of the mass and its motion, ii) the second term is the smoothed-out contribution of the machanical stress and iii) the third term corresponds to the contributions of the correction tensor P_{ij} plus the contribution originated by the smoothing-out of the SS energy momentum tensor through its expansion in terms of the diference h_{ij}. The energy momentum for the LS medium presented here corrects equation (48) given in Noonan's (1984) paper.

It has been conjectured (Ellis, 1984) that the smoothed-out ~~energy momentum~~ stress energy tensor do not necessarily satisfy the usual 'energy conditions' (Hawking and Ellis, 1973 and Wald, 1984). Carfora and Marzuolli in a recent letter to Phys. Rev. (Carfora and Marzuolli, 1984) have concluded that by smoothing-out locally inhomogeneous and anisotropic closed Universe models into the closed FLRW Universes (Λ = 0), the dominant energy condition is not violated; however a more accessible understanding and real application of the approach developed by them needs to done.

The strong energy condition may be expressed in the form (Wald, 1984)

$$T^{ij}\xi_i\xi_j \geq -T/2, \tag{8.37}$$

where ξ_i is any unit timelike vector field. For a perfect fluid eq. (8.37) takes the usual form $\mu + 3p \geq 0$ and $\mu + p \geq 0$.

For the LS energy momentum tensor given in eq. (8.36) the strong energy condition (8.37) takes the form

$$T^{ij}u_i u_j + T/2 = \langle \tilde{\mu}\gamma u^o\Omega \rangle + \langle (u^o)^{-2}N'^{ij}V_iV_j + 1/2\, N'^k{}_k \rangle +$$

$$+ 1/2 \langle \tilde{\mu}\gamma u^o\Omega[\gamma^2 h_{ij}V'^iV'^j - h^k{}_k] \rangle +$$

$$+ 1/4 \langle \tilde{\mu}\gamma\Omega[1/2\,(h^k{}_k)^2 + h^{ij}h_{ij} -$$

$$- \gamma^2 h_{ij}V'^iV'^j(h^k{}_k - 3/2\,\gamma^2 h_{kl}V'^kV'^l) \rangle +$$

$$+ \langle q^i{}_i + (u^o)^{-2}q_{ij}V^iV^j + 1/2\, r^k{}_k[(u^o)^{-2}h_{ij}V^iV^j - 1/2\, h^l{}_l] -$$

$$- 1/2\, r_{ij}[h^{ij} + (u^o)^{-2}V^ih^{jk}V_k] \rangle \geq 0. \qquad (8.38)$$

where we have defined $\Omega = [(u^oV_kV'^k)^2 - 1/2\,(u'^o)^{-2}]$.

Thus, for the strong energy condition to be satisfied for the LS energy momentum, eq. (8.38) must hold for any observer and so in particular along the LS world lines. Although some terms in eq.(8.38) have a definite sign, the overall contribution of the individual terms seems to have no definite sign and so the possibility exists of a violation of the strong energy condition. Despite the fact that we do not have a specific example giving negative energies, (8.38) shows we cannot be confident the strong energy condition will hold in the smoothed-out medium.

8.4 Discussion

We have introduced here the foundations of the smoothing-out theory based on the smoothing-out operator S defined in eq. (8.2). The correction term P_{ij} that must be included in the field equations for the large scale medium and the ~~energy momentum~~ stress energy tensor of the LS medium are given in eqs. (8.27) and (8.36) respectively. It is shown that the SS observer and the LS observer have different perception of what they measure. In particular the proper matter density measured by the SS observer differs from that measured by the LS observer.

The strong energy condition expressed in eq. (8.38) for the LS observer is far too complicated to provide a general statement on its range of validity. It involves, besides the first two terms, the first and second order contribution of the difference h_{ij} and the contribution from the correction term P_{ij} which introduces first and second covariant derivatives of h_{ij}. Thus, it seems more appropriate to consider specific cases while investigating the validity of (8.38).

Equation (8.9) may be interpreted in two different ways: firstly, it can be rewritten in the form

$$g'_{ij} = g_{ij} + h_{ij},$$

which immediately suggests that the whole problem can be treated as a perturbation theory, where h_{ij} is a (small) perturbation to the curved space described by g_{ij}. Thus in this case the gravitational wave solutions to the vacuum ($T'^{ij} = 0$) Einstein field equations can be straightforwardly reproduced by taking $g_{ij} = \eta_{ij}$ (see Isaacson, 1968). The non-vacuum ($T'^{ij} \neq 0$) case in the weak-field slow-motion approximation and Chandrasekhar's post-Newtonian approximation have been recently investigated by Noonan (1984 and 1985). The natural step to be followed is to consider the background metric g_{ij} as the flat FLRW Universe, and the general scalar perturbation for the FLRW Universes which in a synchronous gauge is given by (see Brandenberger et al., 1983)

$$h_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & A\delta_{\alpha\beta} + B_{,\alpha\beta} \end{pmatrix},$$

where $A = A(t, x^{\alpha})$ and $B = B(t, x^{\alpha})$. Thus, the dynamical degrees of freedom for the scalar metric perturbations reside in the two functions A and B.

The second interpretation for the eq. (8.9) is to assume it as a pure definition of the smoothing-out of the SS metric g'_{ij}. The simplest approach to the smoothing-out of the SS metric is by averaging it over a certain region of the space-time. As the smoothing-out here is not due to a physical process, but purely a scale of representation problem over the surfaces t = const., the averaging operator defined in eq. (8.1) can be written in terms of the proper spatial volume instead of the 4-volume.

A possible different way of seeing the smoothing-out problem, based on an averaging process to obtain a Universe model that looks like the FLRW Universes, is by averaging the kinematic quantities, demanding that this average vanish [this may provide a natural minimum length scale beyond which the space-time can be seen as spatially homogeneous and isotropic] and then constructs the metric which corresponds to the smoothed-out quantities of the space-time. Intuitively this is a simple procedure, however the actual computation of the averaged metric is not a simple and well defined matter. In addition, we have no 'a priori' guarantee that the metric obtained by the averaging process used to average the kinematic quantities will in fact lead to a smoothed-out space-time with corresponding vanishing averaged kinematical quantities.

The smoothing-out of local inhomogeneities at different scales of description of the Universe has enlarged the possibility of constructing more realistic Universe models as both virtues - inhomogeneous at a small scale and spatially homogeneous and isotropic at large scale - are merged in a single space-time structure. The implications of such 'Lumpy Universe Models' are still to be investigated. However we hope that the material of this chapter has contributed to clarify the foundations and make the subject more accessible.

REFERENCES

Bianchi, L. (1897). Lezioni sulla Teoria dei Gruppi Continui Finiti di Transformazioni, Spoerri, Pisa.

Birell, N. D. and Davies, P. C. W. (1982), Quantum Fields in Curved Spaces (Cambridge University Press, Cambridge).

Bogoyavlenski, O. and Novikov, S. P. (1973), in Trudy Seminara I. G. Petrovskii, Moscow State University 1, 7 (in russian; engl. transl. in Sel. Math. Sov. 2).

Bonnor, W. B. and Tomimura, N. (1976), Mon. Not. R. Astr. Soc. 175, 85.

Boothby, W. M. (1975), An Introduction to Differentiable Manifolds and Riemannian Geometry (Academic Press, New York).

Brandenberger, R.; Kahn, R. and Press, W. H. (1983), Phys. Rev. D, 28, 1809.

Carfora, M. and Marzuolli, A. (1984), Phys. Rev. Lett. 53, 2445.

Carmeli, M. (1982), Classical fields: General Relativity and Gauge Theory (Wiley & Sons - Interscience).

Choquet-Bruhat, Y.; DeWitt-Morette, C. and Dillard-Bleick, M. (1978), Analysis, Manifolds and Physics (North-Holland).

Cohn, P. M. (1957), Lie Groups (Cambridge University Press, Cambridge).

Collins, C. B. (1977), J. Math. Phys. 18, 2116.

Collins, C. B. and Ellis, G. F. R. (1979), Phys. Reports, 56, 65.

Collins, C. B. and Hawking, S. W. (1973), Astrophys. Journ., 180, 317.

Eisenhart, L. P. (1962), Continuous Groups of Transformations (Dover, New York).

Ellis, G. F. R. (1972), in General Relativity and Cosmology, ed. by R. K. Sachs (Academic Press, New York)

(1973), in Cargese Lectures in Physics, vol VI, ed. by E. Schatzmann (Gordon and Breach, Science Publishers).

(1984), in G.R.G. 10 International Conference Proceedings, ed. by B. Bertotti, F. Felice and A. Pascolini (D. Reidel Publishing Company).

Ellis, G. F. R. and MacCallum, M. A. H. (1969), Commun. Math. Phys., 12, 108.

Ellis, G. F. R. and King, A. R. (1974), Commun. Math. Phys., 38, 119.

Ellis, G. F. R. and Schmidt, B. G. (1977), G. R. G., 8, 915.

Ellis, G. F. R. and Stweart, J. M. (1968), J. Math. Phys., 9, 1072.

Ehlers, J. (1961), Mainz. Akad. Wiss. u Litt., Math.-Nat. Kl., 11.

Estabrook, F.B.; Wahlquist, H.D. and Behr, C.G. (1968), J. Math. Phys., 9, 497.

Grusey, F. (1962), in Relativity, Groups and Topology, ed. C. DeWitt and B. DeWitt (Gordon and Breach, Science Publishers).

Hamermesh, M. (1962), Group Theory and its Application to Physical Problems (Addison-Wesley, London).

Harvey, A. (1979), J. Math. Phys., 20, 251.

Hawking, S. W. and Ellis, G. F. R. (1973), The Large Scale Structure of Space-Time (Cambridge University Press, Cambridge).

Hawking, S. W. and Penrose, R. (1970), Proc. Roy. Soc. Lond. A314, 529.

Heckmann, D. and Schucking, E. (1962), in Gravitation, ed. by L. Witten (Wiley & Sons, New York)

Helgason, S. (1978), Differential Geometry, Lie Groups and Symmetric Spaces (Academic Press, New York).

Hicks, J. (1965), Notes on Differential Geometry (Van Nostrand, Princeton).

Isaacson, R. A. (1968), Phys. Rev. 166, 1263 and 1272.

Jantzen, R. T. (1979), Commun. Math. Phys. 64, 211.

(1983), Ann. Phys. (New York) 145, 378.

(1984), in Cosmology of the Early Universe, ed. by R. Riffini and L. Z. Fang (World Scientific, Singapore).

Kantowiski, R. and Sachs, R. K. (1966), J. Math. Phys. 7, 443.

King, R. and Ellis, G. F. R. (1973), Commun. Math. Phys. 31, 209.

MacCallum, M. A. H. (1973), in Cargese Lectures in Physics, vol. VI, ed. by E. Schatzmann (Gordon and Breach,).

(1979a), in Physics of the Expanding Universe, ed. by M. Demainski (Springer-Verlag, Berlin).

(1979b), in Einstein Centenary Survey, ed. by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge).

Marochnik, L. S., Pelikhov, N. V. and Vereshkov, G. M. (1975), Astrophys. Space Sci. 34, 249 and 281.

Misner, C. W.; Thorne, K. S. and Wheeler, J. A. (1973), Gravitation (W. H. Freeman).

Noonan, T. W. (1984), G. R. G., 16, 1103.

(1985), G. R. G., 17, 535.

Pirani, F. A. E. (1956), Phys. Rev. 105, 1089.

Rosquist, K. (1983), Phys. Letters, 97A, 145.

(1984), Class. Quantum Grav., 1, 81.

Ryan, M. P. Jr. and Shepley, L. C. (1975), Homogeneous Relativistic Cosmologies (Princeton University Press, Princeton).

Sachs, R. K. and Wu, H. (1977), General Relativity for Mathematicians (Springer-Verlag, New York Inc.).

Sagle, A. A. and Walde, R. E. (1973), Introduction to Lie Groups and Lie Algebras (Academic Press, New York).

Schouten, J. A. (1954), Ricci Calculus (Springer-Verlag, Berlin).

Siklos, S. T. C. (1978), Commun. Math. Phys., 58, 255.

(1980), Phys. Letters, 67A, 19.

Taub, A. H. (1951), Ann. Math. 53, 472.

Thorne, K. S. (1967), Astrophys. Journ., 148, 51.

Tipler, F. J. (1977), Phys. Letters, 64A, 8.

Tipler, F. J.; Clarke, J. C. and Ellis, G. F. R. (1980), in General Realtivity and Gravitation, ed. by A. Held (Plenum Press, New York).

Trautman, A. (1964), in Lectures on General Relativity ed. by S. Deser and K.W. Ford (Englewood Cliffs, NJ: Prentice-Hall).

von Westenholz, C. (1978), Differential Forms in Mathematical Physics (North-Holland)

Wainwright, J. (1984), G. R. G., 16, 1103.

Wald, R. M. (1984), General Relativity (University of Chicago Press, Chicago).

Weber,J.(1962), General Relativity and Gravitational Waves (Wiley-Interscience).

APPENDIX

1-Parameter Families of Vacuum Solutions for Bianchi types IV, VI_h and VII_h Universes

W. L. Roque and M. J. Jaklitsch

Department of Applied Mathematics

University of Cape Town

Rondebosch, 7700

South Africa

Abstract

Following the compactification procedure developed by Rosquist (1984) we show that the equilibrium subsystems for Bianchi types VI_o and VII_o do not admit vacuum solutions. 1-parameter families of vacuum solutions for Bianchi types IV, VI_h and VII_h are presented. The dust and radiation solutions of Collins (1971) are reobtained as solutions from the Taub equilibrium subsystem given by Rosquist.

1. Introduction

For a long time controversy has surrounded the ADM based Hamiltonian formulation of the dynamics of spatially homogeneous cosmologies. Problems arise due to the non-commutativity of the variational principle with imposition of symmetries, i.e. the field equations obtained from a Lagrangian with symmetries imposed in it do not necessarily agree with those obtained by imposing the symmetries on the field equations obtained from the general Lagrangian (MacCallum, M.A.H and Taub, A.H., 1972). Recently, Jantzen (1979, 1983) has revived the subject and has shown that the correct field equations for spatially homogeneous cosmologies can be obtained from a Hamiltonian formulation of General Relativity by introducing a non-potential term into the equations of motion for the components of the 3-metric (Jantzen, 1979).

The virtue of the Jantzen approach lies in the use of the time dependent automorphism group of the Bianchi Lie algebras in order to diagonalize the metric and simplify the field equations. Bianchi metrics can always be diagonalised by use of the automorphism group and in some cases a further simplification may be possible (Jantzen, 1979, Roque and Ellis, 1985). In the Jantzen formalism the Einstein field equations are given by a set of 12 first order ordinary differential equations plus 4 constraint equations. By making use of the constraint equations the matter variables for a perfect fluid may be written in terms of geometrical variables resulting in a 12 dimensional autonomous system.

Based on Jantzen's Hamiltonian formulation, Rosquist (1984) has worked out the field equations for the Bianchi type VI_h Universe models. He shows [that on using the scale invariance (Jantzen, 1983) of the Einstein field equations and compactification of the gravitational phase space] the field equations can be reduced effectively to a 7-dimensional autonomous system. When looking for the simplest cases which admit rotation, the 7-dimensional system reduces to a 4-dimensional Taub subsystem. An exact solution for the Taub subsystem has been found by Rosquist (1984) which corresponds to an expanding and rotating Bianchi type VI_0 Universe filled with radiation. This solution has the intriguing feature of constant tilt angle.

In an investigation of the Taub equilibrium subsystem given by Rosquist (1984), we have found three other solutions which correspond to vacuum, dust and radiation solutions for Bianchi type VI_0 Universes.

The vacuum solution that we have found from the Taub equilibrium subsystem provided by Rosquist (1984) has been found to be inconsistent with the results of Roque and Ellis (1985) (see Roque, 1985, for details), who show that, for the vacuum class A Bianchi types VI_0 and VII_0, the special automorphism group parameters must be constant rather than functions of time. The vacuum solution found by us from the Taub equilibrium subsystem contradicted this fact. The problem occurs when, as in Rosquist's method the Hamiltonian and momentum constraint equations are used to replace all the matter variables by the geometrical variables; for, in the vacuum case (since there is no matter present) this approach does not make sense. Instead, the constraints must be solved first, imposing conditions on some of the geometrical variables.

In this paper we will examine the compactified vacuum field equations for class B Bianchi Universes. 1-parameter families of vacuum solutions for Bianchi types IV, VI_h and VII_h Universes are presented. For Bianchi type V, a vacuum solution has been found from the equilibrium system corresponding to an open ($K = -1$) vacuum FRW Universe. In particular, we show that the equilibrium subsystem for the vacuum Bianchi Type VI_0 does not admit a solution. A similar result also holds for the compactified equilibrium system of field equations for vacuum Bianchi type VII_0. This supports the result of Roque (1985) that any solution of the equilibrium subsystem for Bianchi types VI_o and VII_o must have constant special automorphism group parameters. The dust and radiation solutions for Bianchi type VI_o presented here are not new; they belong to the family of solutions previously found for Bianchi type VI_o by Collins (1971). However as they have been reobtained in an approach different to that used by Collins, we will briefly describe them for sake of completeness.

The structure of the paper is as follows: section 2 gives an overview of the Rosquist (1984) Taub subsystem for Bianchi type VI_h and defines some quantities in terms of his notation. [We will follow his notation throughout this paper]. Section 3 gives the changes that must be made in order to obtain the correct compactified field equations for vacuum Bianchi type VI_h and we show that there is no solution for the vacuum equilibrium subsystem of type VI_o. The vacuum solutions for the Bianchi types IV, VI_h and VII_h Universes are also presented. Section 4 gives the dust and radiation solutions. Section 5 carries some concluding remarks.

2. The Taub subsystem

In order to introduce some notation which will be used in the following sections, we review the Taub subsystem given by Rosquist (1984). This 4-dimensional subsystem for Bianchi type VI_h is characterized by ($v = 1$, $s_- = 0$, $r_1 = \pm r_2$); the equations are

$$\dot{s}_+ = B_+ - B_o s_+, \tag{1.a}$$

$$\dot{r}_a = C_a - B_o r_a, \; (a = 2, 3), \tag{1.b}$$

$$\dot{u} = -2u(B_o + 2(1 - s_+)). \tag{1.c}$$

A solution for the equilibrium subsystem of (1) (obtained by setting all derivatives zero, i.e., $\dot{s}_+ = \dot{r}_a = \dot{u} = 0$, $u \neq 0$), has been found by Rosquist (1984), (note a misprint in the solution presented there) for a radiation filled Bianchi type VI_0 universe. His correct solution is given by:

$$s_+ = 1/2, \tag{2.a}$$

$$r_2 = (-3 + \alpha)/12\sqrt{2}, \tag{2.b}$$

$$r_3 = (1 + \alpha)/32\sqrt{3}, \tag{2.c}$$

$$u = (283 - 21\alpha)/24576, \tag{2.d}$$

where $\alpha = \sqrt{33}$. The metric for this solution is given explicitly in Rosquist, (1983).

The energy-density ρ in the reduced phase space is given by

$$\rho = 3\hat{\rho}/2N^2, \quad \hat{\rho} = (\gamma - 1)^{-1}((\gamma - 2)\bar{E} + \gamma\sqrt{\bar{F}}), \tag{3.a}$$

with the limiting case

$$\hat{\rho} \to 2(\bar{E}^2 - \bar{\Sigma})/\bar{E} \quad \text{as } \gamma \to 1. \tag{3.b}$$

The lapse function $N(t)$ in equation (3.a) is given by

$$N = 12\, g^{1/2} \lambda^2, \tag{4}$$

where $g = e^{6\beta^0}$ and $\lambda = p_0^{-1/2}$.

The expansion scalar of the normal congruence is determined by

$$\theta = -\,(1/4)g^{-1/2}p_0. \tag{5}$$

Choosing $p_0 > 0$ (< 0) corresponds to contraction (expansion) of the normal congruence. We choose p_0 to be negative in order to have $\theta > 0$, i.e. time directed away from the singularity.

The length scalar l and the shear scalar σ are defined by (Ellis, 1971):

$$\dot{l}/l = \theta/3, \tag{6}$$

$$\sigma^2 = 1/2\ \sigma_{\alpha\beta}\sigma^{\alpha\beta}. \tag{7}$$

The tilt angle, β, in the reduced phase space is given by

$$\sinh^2\beta = (2\bar{E} - \bar{D})/(\gamma\bar{D} - 2\bar{E}), \tag{8.a}$$

valid for $1 < \gamma \leq 2$, with a limiting case for $\gamma = 1$ given by

$$\sinh^2\beta = \bar{\Sigma}/(\bar{E}^2 - \bar{\Sigma}) \tag{8.b}$$

It turns out that β is constant for the solution (2). Actually any tilted solution of the equilibrium subsystem will have constant tilt angle with the exceptional cases of $\gamma\bar{D} \to 2\sqrt{\bar{\Sigma}}$, which corresponds to whimper singularities, $\beta \to \infty$ (Ellis and King, 1974).

3. The vacuum solutions for Bianchi types IV, V, VI_h and VII_h

The compactification procedure developed by Rosquist cannot be applied directly to the vacuum case. A solution to the equilibrium system of eqs. (1) has been found, and is given by

$$s_+ = 5/8, \quad r_2 = \sqrt{2}/8, \quad r_3 = \sqrt{3}/8, \quad u = 1/192, \tag{9}$$

which has (from eq. 8.a) $\beta = 0$. However this does not correspond to a solution of the Einstein field equations. As described in the introduction, the failure of this solution is due to the fact that in the vacuum case the momentum constraints are identically zero, and so they cannot be used to replace the matter variables by the geometrical variables as has been done by Rosquist.

Indeed, in the vacuum case there is no matter to consider. Thus, when considering the Rosquist compactified system for the vacuum case of Bianchi type VI_h (no restriction to the Taub subsystem is needed) the following amendments must be made in order to get the correct compactified set of equations:

$$B_o = -3\bar{K}/2 - \bar{W}, \tag{10.a}$$

$$\hat{H}_1 = \hat{H}_2 = \hat{H}_3 = 0, \tag{10.b}$$

which imply $\hat{\Sigma}_a = \bar{\Sigma} = \bar{\Sigma}_i = 0$ where ($a = 1$, 2, 3 and $i = +, -$). With these amendments, the equations given by Rosquist (1984) become the correct compactified system for vacuum Bianchi type VI_h. When looking for equilibrium solutions the momentum constraints, $\hat{H}_a = 0$, should first be solved, imposing conditions on the geometric variables r_a, and then these conditions are applied to the coupled 7-dimensional autonomous subsystem (Rosquist, 1984, eq. 3.7 with derivatives set equal to zero) which now must be solved in order to obtain explict vacuum solutions.

Following Rosquist (1984), the compactified field equations for Bianchi types IV, V and VII_h have been obtained. They are given in the Appendix. As has been pointed out by Rosquist (1984), the value of $v = 1$ corresponds to a singularity in the parametrization of the metric for types V and VII_h and so we restrict v to lie in the range of $0 < v < 1$ for these types.

For the class A Bianchi types VI_o and VII_o, the conditions $\hat{H}_a = 0$ imply that $r_a = 0$. Thus the equilibrium conditions $\dot{r}_a = 0$ are trivially satisfied. The condition $\dot{v} = 0$ implies $s_- = 0$, which when applied to the condition $\dot{s}_- = 0$ requires that $B_- = 0$. However, for Bianchi type VI_o, $B_- = 0$ requires that $u = 0$, while for Bianchi type VII_o, $B_- = 0$ forces $v = 1$. Therefore, in both cases the equilibrium conditon forces the geometric variables u and v to be out of their range of validity (i.e., in the case of type VI_o, $u = 0$ contradicts the previous condition $u \neq 0$ and in the case of type VII_0, $v = 1$ contradicts the condition $v < 1$). So we conclude that there are no vacuum solutions for the equilibrium subsystems for Bianchi types VI_0 and VII_0.

The situation in the class B case is somewhat different. Now the constraints $\hat{H}_1 = \hat{H}_2 = 0$ are satisfied by setting $r_1 = r_2 = 0$, while the condition $\hat{H}_3 = 0$ gives a relation between the geometric variables r_3 and s_+. The equations $\dot{r}_3 = 0$ and $\dot{s}_+ = 0$ lead to the same expression for B_0 in terms of the variables r_3 and v. When this expression is substituted into the equation obtained from $\dot{u} = 0$, the result is an equation relating r_3 to v. Thus the equilibrium system can be solved explicitly once a value of v has been assumed.

If we consider v as a parameter, the explicit relations between the geometric variables r_3, u, and s_+ and the parameter v for Bianchi types IV, VI_h and VII_h are given by

$$r_3 = - f(2\sqrt{3}av)^{-1} [1 + f^2/12a^2v^2]^{-1}, \tag{11.a}$$

$$u = (12av + f^2/av)^{-2}, \tag{11.b}$$

$$s_+ = - fr_3(2\sqrt{3}av)^{-1}, \tag{11.c}$$

where $B_0 = 2(s_+ - 1)$ and $f = (v^2, v^2 + 1, v^2 - 1)$ for types (IV, VI_h and VII_h) respectively. They are valid for $0 < v \leq 1$ for types IV and VI_h and $0 < v < 1$ for type VII_h and $h \neq 0$ for Bianchi types VI_h, VII_h and with $a = 1$ for type IV.

The explicit forms for the metric corresponding to each of these solutions are given in the invariant basis (Jantzen, 1979) by

Bianchi type IV:

$$ds^2 = - dt^2 + vt^{(-B_o/3+B_o)} (\omega^1)^2 + v^{-1}t^{(-B_o/3+B_o)}[1 + (v\theta^3)^2] (\omega^2)^2 +$$

$$+ v^{-2}u^{-1}(1/4 + B_o/12)^2 t^2 (\omega^3)^2 - v\theta^3 t^{(-B_o/3+B_o)} \omega^1\omega^2, \tag{12}$$

Types VI_h and VII_h:

$$ds^2 = - dt^2 + v^{-1}[v^2c^2 + s^2] t^{(-B_o/3+B_o)} (\omega^1)^2 +$$

$$+ v^{-1}[v^2s^2 + c^2]t^{(-B_o/3+B_o)} (\omega^2)^2 + v^{-2}u^{-1}(1/4 + B_o/12)^2 t^2 (\omega^3)^2 -$$

$$- csv^{-1}ft^{(-B_o/3+B_o)} \omega^1\omega^2, \tag{13}$$

where the automorphism group parameter θ^3 is given by

$$\theta^3 = - B_o[2a(3 + B_o)]^{-1} \ln t. \tag{14}$$

and for brevity we have defined $(c, s) \equiv (\cosh \theta^3, \sinh \theta^3)$ for Bianchi type VI_h and $(c, s) \equiv (\cos \theta^3, \sin \theta^3)$ for Bianchi type VII_h, with $(h \neq 0)$ and the parameter $a = 1$ for type IV.

The kinematic quantities for these solutions - expansion and shear - are

$$\theta = 3(3 + B_o)^{-1} t^{-1}, \tag{15.a}$$

$$\sigma^2 = 3/4 [(2 + B_o)/(3 + B_o)]^2 t^{-2}, \tag{15.b}$$

where $B_o = B_o(v)$ and restricted to $B_o \geq - 2$.

The Bianchi type V vacuum solution obtained from the equilibrium system has the metric

$$ds^2 = -dt^2 + t^2[e^{-2z}(dx^2 + dy^2) + dz^2], \tag{16}$$

where the expansion is given by $\theta = 3t^{-1}$ and the shear vanishes.

4. The Dust and Radiation solutions

A dust solution to the Taub equilibrium subsystem (1) for Bianchi type VI_0 universe is given by:

$$s_+ = 1/4, \tag{17.a}$$

$$r_1 = r_2 = r_3 = 0, \tag{17.b}$$

$$u = 1/256. \tag{17.c}$$

It turns out that the tilt angle defined in eq.(8.b) vanishes, $\beta = 0$. This implies that the fluid flow lines are hypersurface orthogonal. The expansion, shear, length scalar and energy density are given by:

$$\theta = 2\, t^{-1}, \tag{18.a}$$

$$\sigma = 1/2\sqrt{3}\, t^{-1}, \tag{18.b}$$

$$l = l_0\, t^{2/3},\ l_0 = \text{const.} \neq 0, \tag{18.c}$$

$$\rho = t^{-2} \tag{18.d}$$

In the invariant basis of type VI_0 (Jantzen, 1979) the line element of this solution is given by:

$$ds^2 = -dt^2 + t(\omega^1)^2 + t(\omega^2)^2 + 4t^2(\omega^3)^2. \tag{19}$$

For Bianchi VI_0 the invariant basis is given in terms of the coordinate basis as

$$\omega^1 = \cosh z\, dx + \sinh z\, dy \tag{20.a}$$

$$\omega^2 = \sinh z\, dx + \cosh z\, dy \tag{20.b}$$

$$\omega^3 = dz. \tag{20.c}$$

Thus in the coordinate basis the line element is

$$ds^2 = -dt^2 + t\cosh 2z\, dx^2 + t\cosh 2z\, dy^2 + 4t^2dz^2 + 2t\sinh 2z\, dxdy \qquad (21)$$

A radiation solution of eqs. (1) for Bianchi type VI_o is given by:

$$s_+ = 1/2, \qquad (22.a)$$

$$r_1 = r_2 = r_3 = 0, \qquad (22.b)$$

$$u = 1/192 \qquad (22.c)$$

Again this represents a non-tilted solution with fluid 4-velocity u^a = (1,0,0,0). The expansion, shear, length scalar and energy density are

$$\theta = 3/2\ t^{-1} \qquad (23.a)$$

$$\sigma = \sqrt{3}/4\ t^{-1} \qquad (23.b)$$

$$l = l_o\ t^{1/2},\ l_o = \text{const.} \neq 0, \qquad (23.c)$$

$$\rho = 3/8\ t^{-2} \qquad (23.d)$$

The line element in the invariant basis is given by:

$$ds^2 = -dt^2 + t^{1/2}(\omega^1)^2 + t^{1/2}(\omega^2)^2 + (16/3)t^2(\omega^3)^2 \qquad (24.a)$$

and in the coordinate basis by:

$$ds^2 = -dt^2 + t^{1/2}\cosh 2z\, dx^2 + t^{1/2}\cosh 2z\, dy^2 + (16/3)t^2\, dz^2 + 2t^{1/2}dxdy. \qquad (24.b)$$

5. Conclusion

We have presented here a 1-parameter family of vacuum solutions for Bianchi types IV, VI_h and VII_h Universes (for types VI_h and VII_h h gives, in fact, a two parameter family). As far as we have been able to establish the equivalence, or otherwise, between our solutions and previously found exact solutions for these Bianchi types listed in Kramer et al. (1980), they are new. The solutions have been obtained as equilibrium solutions for the compactified vacuum field equations for these models. The vacuum solution for the equilibrium system of Bianchi type V corresponds to an open (k = -1) vacuum FRW Universe. The metrics are given in terms of the compactified geometric variable v, which plays the role of a bounded parameter. An explanation for there being an infinite family of solutions for each type can be given by considering the shear (eq. 15b). Since the shear is a function of the parameter v, by varying v we vary the shear and consequently the expansion rate. Hence the shear is to some extent arbitrary in much the same way as in the vacuum Bianchi type I Universe (Heckmann and Schucking, 1962).

We have shown that the equilibrium subsystems for Bianchi types VI_o and VII_o admit no vacuum solutions. In addition, we have also presented non-tilted dust and radiation solutions for type VI_o, which were obtained as equilibrium solutions to the Taub subsystem given in section 2. These two solutions are not new; they have been previously found by Collins (1971) (see Kramer et al. 1980 with correct addendum).

Appendix

The compactified field equations for Bianchi types IV, V and VII_h are given by (we follow the same definitions as Rosquist, 1984):

$$\dot{w} = -1/2\,(B_o + 2)w\,, \qquad \dot{p}_o = p_o B_o\,,$$

$$\dot{s}_+ = B_+ - s_+ B_o\,, \qquad \dot{r}_a = C_a - r_a B_o,$$

$$\dot{u} = -2u[B_o + 2(1 - s_+ + \sqrt{3}s_-)]\,, \qquad \dot{v} = 2\sqrt{3}vs_-,$$

where for Bianchi type IV we have

$$B_o = -3/2\,(2 - \gamma)\bar{K} - 1/2\,(3\gamma - 2)\bar{W} + (4 - 3\gamma)\bar{\Sigma}(\gamma\bar{D})^{-1},$$

$$B_+ = 3[r_1^{\,2} + r_2^{\,2}] - 24uv^4 + \bar{\Sigma}_+(\gamma\bar{D})^{-1},$$

$$B_- = \sqrt{3}[r_1^{\,2} - r_2^{\,2} + 2r_3^{\,2}] - 24\sqrt{3}uv^4 + \bar{\Sigma}_-(\gamma\bar{D})^{-1},$$

$$C_1 = -(3s_+ + \sqrt{3}s_-)r_1 + \hat{\Sigma}_1(\gamma\bar{D})^{-1},$$

$$C_2 = 2\sqrt{3}r_1 r_3 - (3s_+ - \sqrt{3}s_-)r_2 + \hat{\Sigma}_2(\gamma\bar{D})^{-1},$$

$$C_3 = -2\sqrt{3}r_1 r_2 + 48\sqrt{3}uv^3 - 2\sqrt{3}s_- r_3 + \hat{\Sigma}_3(\gamma\bar{D})^{-1},$$

and

$$\bar{K} = 1 - s_+^{\,2} - s_-^{\,2} - r_1^{\,2} - r_2^{\,2} - r_3^{\,2}\,, \qquad \bar{W} = 12uv^2[12 + v^2],$$

$$\bar{\Sigma} = 48u[\hat{H}_1^{\,2} + v^2\hat{H}_2^{\,2} + \hat{H}_3^{\,2}], \qquad \hat{H}_1 = 3vr_1,$$

$$\bar{\Sigma}_+ = 48u[\hat{H}_1^{\,2} + v^2\hat{H}_2^{\,2} - 2\hat{H}_3^{\,2}], \qquad \hat{H}_2 = -vr_1 + 3r_2,$$

$$\Sigma_- = 48\sqrt{3}u[\hat{H}_1{}^2 - v^2\hat{H}_2{}^2], \qquad \hat{H}_3 = -2\sqrt{3}vs_+ - v^2r_3,$$

$$\hat{\Sigma}_1 = 96\sqrt{3}u\hat{H}_1\hat{H}_3, \quad \hat{\Sigma}_2 = 96\sqrt{3}uv\hat{H}_2\hat{H}_3, \quad \hat{\Sigma}_3 = -96\sqrt{3}uv\hat{H}_1\hat{H}_2.$$

and for Bianchi types V and VII_h we have

$$B_o = -3/2\,(2-\gamma)\bar{K} - 1/2\,(3\gamma-2)\bar{W} + (4-3\gamma)\bar{\Sigma}(\gamma\bar{D})^{-1},$$

$$B_+ = 3[r_1{}^2 + r_2{}^2] - [24uf^2]_{VII_h} + \bar{\Sigma}_+(\gamma\bar{D})^{-1},$$

$$B_- = \sqrt{3}[r_1{}^2 - r_2{}^2 + 2(1+v^2)f^{-1}r_3{}^2] - [24\sqrt{3}u(1+v^2)f]_{VII_h} + \bar{\Sigma}_-(\gamma\bar{D})^{-1},$$

$$C_1 = -2\sqrt{3}f^{-1}r_2r_3 - [3s_+ + \sqrt{3}s_-]r_1 + \hat{\Sigma}_1(\gamma\bar{D})^{-1},$$

$$C_2 = 2\sqrt{3}v^2f^{-1}r_1r_3 - [3s_+ - \sqrt{3}s_-]r_2 + \hat{\Sigma}_2(\gamma\bar{D})^{-1},$$

$$C_3 = -2\sqrt{3}r_1r_2 + [48\sqrt{3}auvf]_{VII_h} - 2\sqrt{3}(1+v^2)fs_-r_3 + \hat{\Sigma}_3(\gamma\bar{D})^{-1},$$

With

$$\bar{K} = 1 - s_+{}^2 - s_-{}^2 - r_1{}^2 - r_2{}^2 - r_3{}^2, \qquad \bar{W} = 12u([f^2]_{VII_h} + 12a^2v^2],$$

$$\bar{\Sigma} = 48u[\hat{H}_1{}^2 + v^2\hat{H}_2{}^2 + \hat{H}_3{}^2], \qquad \hat{H}_1 = 3avr_1 + [r_2]_{VII_h},$$

$$\bar{\Sigma}_+ = 48u[\hat{H}_1{}^2 + v^2\hat{H}_2{}^2 - 2\hat{H}_3{}^2], \qquad \hat{H}_2 = [-vr_1]_{VII_h} + 3ar_2,$$

$$\bar{\Sigma}_- = 48\sqrt{3}[\hat{H}_1{}^2 - v^2\hat{H}_2{}^2], \qquad \hat{H}_3 = -2\sqrt{3}avs_+ - [fr_3]_{VII_h},$$

$$\hat{\Sigma}_1 = 96\sqrt{3}u\hat{H}_1\hat{H}_2, \quad \hat{\Sigma}_2 = 96\sqrt{3}uv\hat{H}_2\hat{H}_3, \quad \hat{\Sigma}_3 = -96\sqrt{3}uv\hat{H}_1\hat{H}_2 \quad \text{and} \quad f = (v^2 - 1).$$

For Bianchi type V we have $a = 1$ and the terms $[\ ]_{VII_h}$ are set equal to zero. For types VI_h and VII_h, h is a parameter with range $-\infty \leq h \leq \infty$. [In order to obtain the correct compactified equations for the vacuum cases of types IV, VI_h and VII_h, the amendments (10) must be made to the general equations provided in this appendix.

Acknowledgments:

We would like to thank Profs. G.F.R.Ellis and D.R. Matravers for helpful discussions and critical comments. We are in debt to Prof. W. B. Bonnor for discussions on an earlier version of this paper and to Dr. M. A. H. MacCallum for correspondence.

References:

Collins, B. B. (1971), Commun. Math. Phys., 23, 137.

Ellis, G.F.R. (1971) in 'General Relativity and Cosmology', ed. R.K. Sachs. (Academic, New York).

Ellis, G. F. R and King, R. (1974), Commun. Math. Phys., 38, 119

Heckmann, O. and Schucking, E. (1962), in Gravitation ed. by L. Witten (Wiley, New York)

Jantzen, R.T. (1979), Commun. Math. Phys. 64, 211

- (1983), Ann. Phys. 145, 378

Kramer, D.; Stephani, H.; Herlt, E. and MacCallum, M. A. H. (1980), Exact Solutions of the Einstein Field Equations (Cambridge University Press, Cambridge).

MacCallum, M.A.H. and Taub, A.H. (1972), Commun. Math. Phys. 25,173

Roque, W.L. and Ellis, G.F.R. (1985) in 'Axisymmetric Systems, Galaxies and Relativity', ed. M.A.H. MacCallum, (Cambridge University Press, Cambridge)

Roque, W. L. (1985), Ph.D. Thesis (University of Cape Town)

Rosquist, K. (1983), Phys. Lett. 97A, 145

- (1984), Class. Quantum Grav. 1, 81